Magnetic Dirac operators with Coulomb-type perturbations

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CIRM, July 2008

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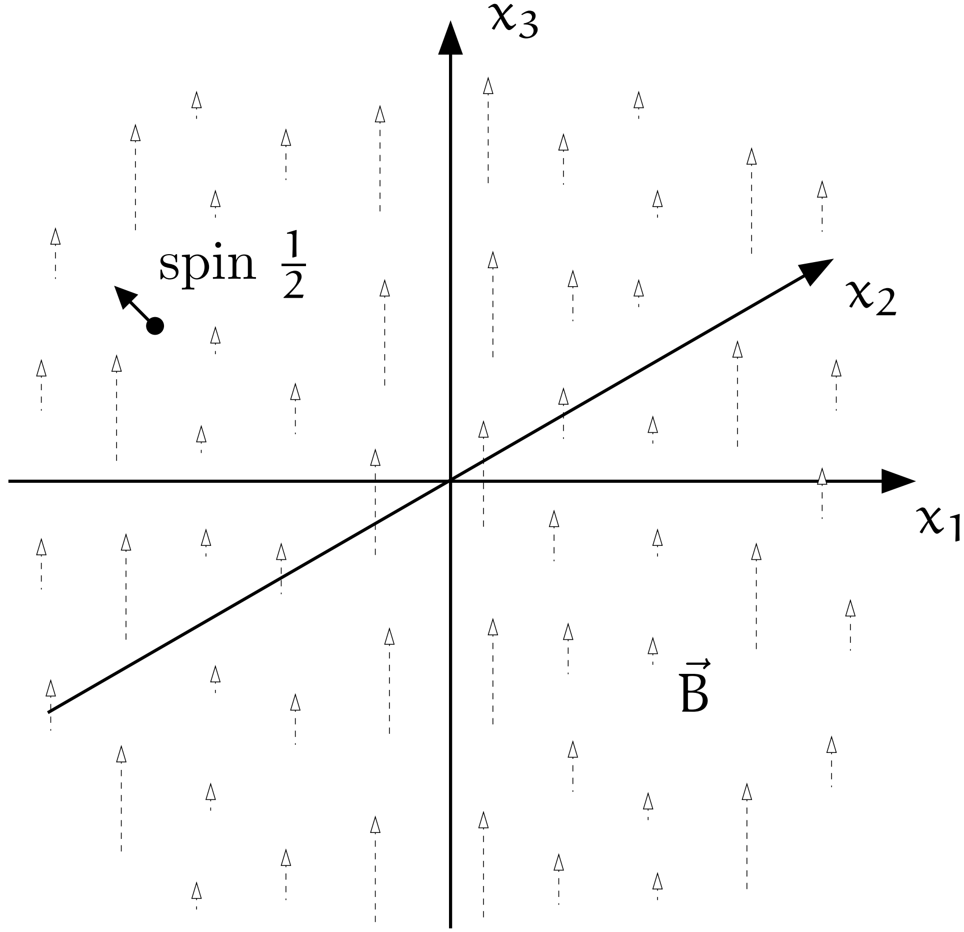
1 Set-up



Relativistic spin-$\frac{1}{2}$ particle in presence of a magnetic field of constant direction $\vec{B}(\vec{x}) \equiv (0, 0, B(x_1, x_2))$, with $B \in L^\infty_{\text{loc}}(\mathbb{R}^2; \mathbb{R})$.

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Associated Dirac operator in $\mathcal{H} := L^2(\mathbb{R}^3; \mathbb{C}^4)$:

$$H_0 := \alpha_1 \Pi_1 + \alpha_2 \Pi_2 + \alpha_3 P_3 + \beta m,$$

where $\beta \equiv \alpha_0$, α_1, α_2, α_3 are the Dirac-Pauli matrices, $m > 0$ is the mass of the particule, and $\Pi_j = P_j - a_j \equiv -i\partial_j - a_j$ are the canonical momenta.

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The vector potential $\vec{a} = (a_1, a_2, 0)$ is chosen such that $a_1, a_2 \in L^\infty_{\text{loc}}(\mathbb{R}^2; \mathbb{R})$ and $B = \partial_1 a_2 - \partial_2 a_1$.

2 Fibered structure of H_0

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The spectrum of H_0 satisfies

$$\sigma(H_0) = \big(-\infty, -\inf |\sigma^0_{\rm sym}| \big] \cup \big[\inf |\sigma^0_{\rm sym}|, \infty \big),$$

where $\sigma^0_{\rm sym} := \sigma(H^0) \cup \sigma(-H^0)$.

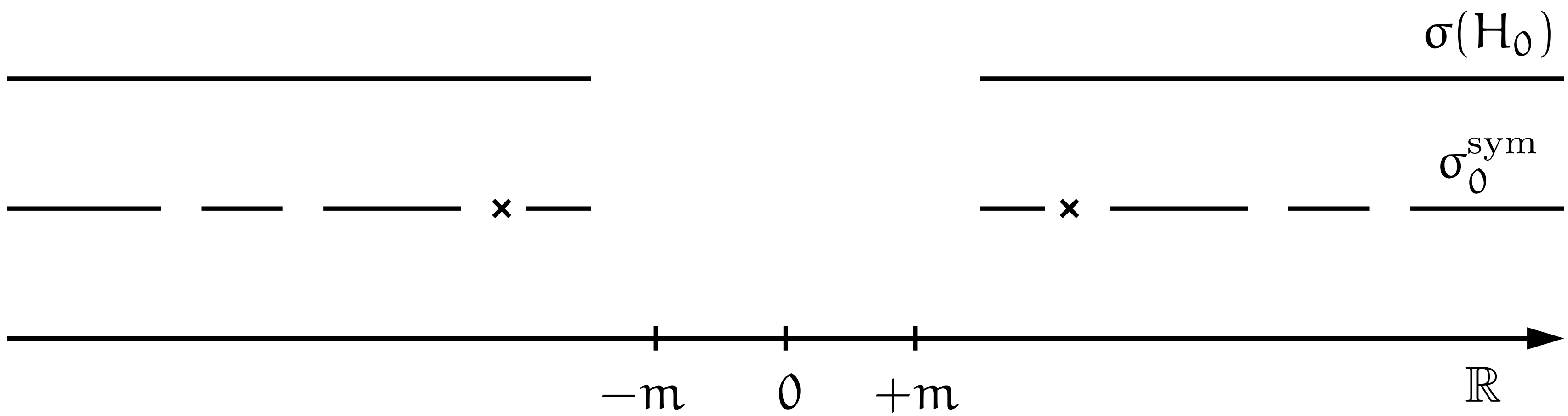

Figure 1: σ^0_{sym} and $\sigma(H_0)$ for some $B \neq$ Const.

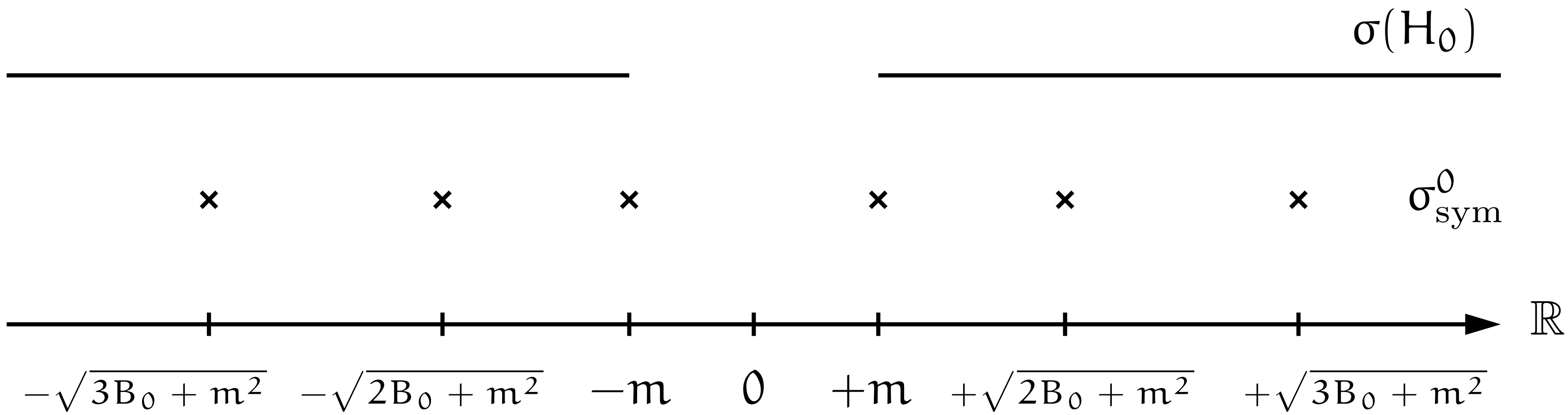

Figure 2: If $B(x_1, x_2) = B_0 > 0$, then

$$\sigma^0_{\text{sym}} = \{\pm\sqrt{2nB_0 + m^2} \mid n = 0, 1, 2, \dots\}$$

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The perturbation $\mathsf{V} : \mathbb{R}^3 \to \mathcal{B}_{\mathrm{h}}(\mathbb{C}^4)$ can be written as $\mathsf{V} = \mathsf{V}_{\mathrm{Coulomb}} + \mathsf{V}_{\mathsf{L}^3} + \mathsf{V}_{\infty}$, where:

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- $V_\infty \in L^\infty_0(\mathbb{R}^3; \mathcal{B}_h(\mathbb{C}^4))$, and $V_\infty = V_{\text{short-range}} + V_{\text{long-range}}$ with decay rate only imposed along x_3.

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5. The limits $\lim_{\varepsilon \searrow 0} \left\langle \psi, (H - \lambda \mp i\varepsilon)^{-1} \psi \right\rangle$ *exist for each* $\psi \in \mathcal{D}(\langle Q_3 \rangle^{1/2+\delta})$, $\delta > 0$, *uniformly in* λ *on each compact subset of* $\mathbb{R} \setminus \{\sigma^0_{\mathrm{sym}} \cup \sigma_{\mathrm{pp}}(H)\}$.

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(In fact the limiting absorption principle is expressed in terms of an interpolation space $(\mathcal{D}(A), \mathcal{H})_{1/2,1} \supset \mathcal{D}(\langle Q_3 \rangle^{1/2+\delta})$ *for some conjugate operator* A.*)*

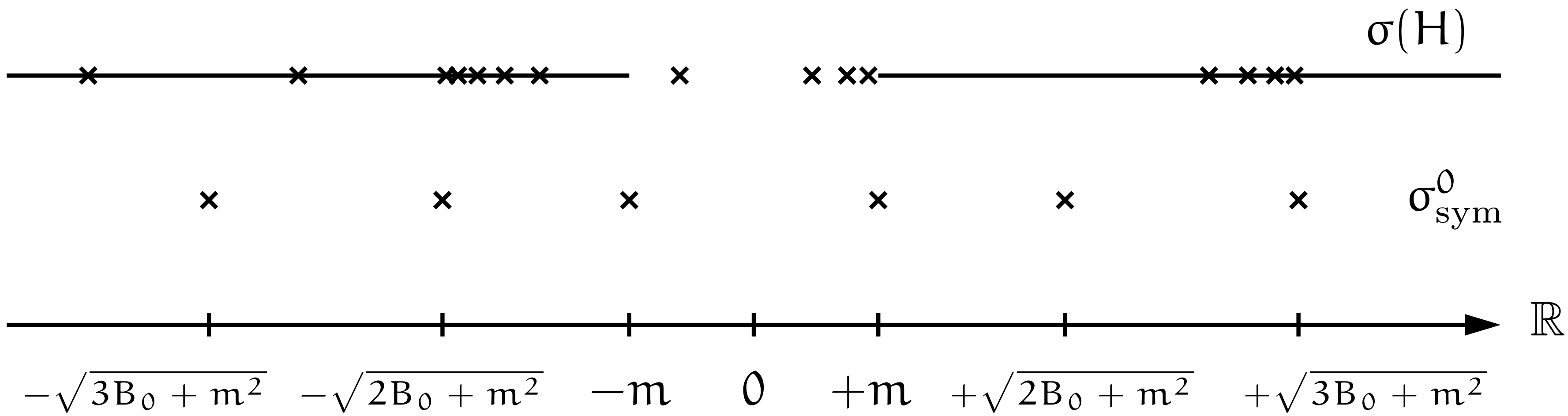

Figure 3: Example of $\sigma(H)$ for $B(x_1, x_2) = B_0 > 0$

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(ii) We prove the selfadjointness of H defined as a form sum by using results of [Nenciu76], [Nenciu77], and [BoutetPurice94] on selfadjointness for Dirac operators.

(iii) We extend to H the Mourre estimate for H_0 by using the commutators methods of [GeorgescuMăntoiu01] for strong local singularities of Dirac operators.

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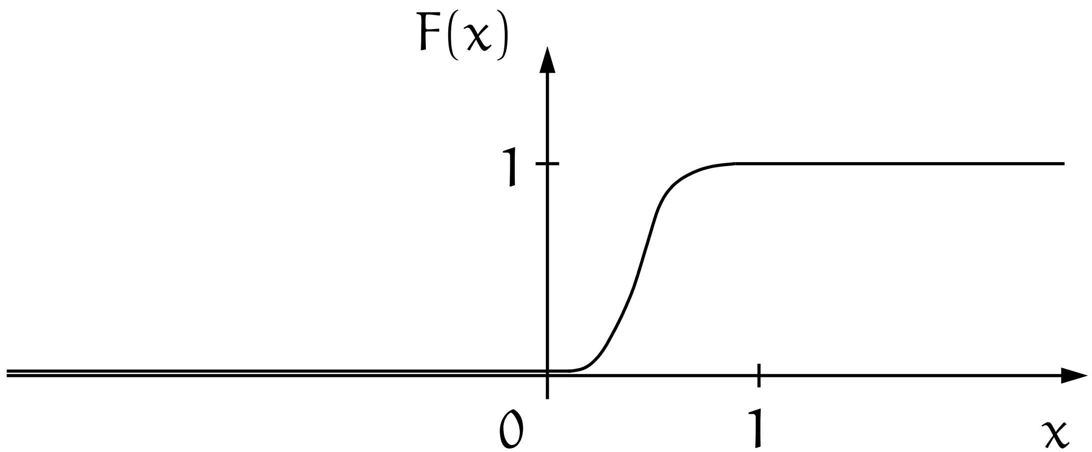
Drawback: `The matrix structure of` A_0 `leads to unnatural conditions (involving commutators with the` α_j`'s) when one treats long-range perturbations` $V_{\text{long-range}}$ `of` H_0`.`

Thus we use the scalar conjugate operator

$$A := \frac{1}{2}[Q_3 F(P_3) + F(P_3) Q_3],$$

where



With this choice we get for $J \subset (0, \infty)$ bounded and $\mathcal{F}_3$ the Fourier transform along x_3:

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&= -\tfrac{1}{2}E_0(J)\mathcal{F}_3^{-1}\int_{\mathbb{R}}^{\oplus} d\xi \left[\Big((\alpha_1\Pi_1+\alpha_2\Pi_2+\beta m)^2+\xi^2\Big)^{1/2}, \partial_\xi F(\xi)+F(\xi)\partial_\xi\right]\mathcal{F}_3 E_0(J) \\
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$$= E_0(J)F(P_3)P_3|H_0|^{-1}E_0(J)$$

$\Longrightarrow$ `It is sufficient to prove the positivity of the bounded operator` $F(P_3)P_3|H_0|^{-1}$ `when localized on` $\sigma(H_0)\setminus\sigma^0_{\text{sym}}$.

Using $\mathcal{F}_3$ and the formula

$$\sigma\big[H_0(\xi)^2\big] = \sigma\big[H_0(0)^2 + \xi^2\big] = (\sigma_0^{\mathrm{sym}})^2 + \xi^2,$$

we deduce the Mourre estimate

$$\begin{aligned}\lim_{\varepsilon\searrow 0}\sup\big\{a\in\mathbb{R}\mid E_0(\lambda;\varepsilon)i[H_0,A]E_0(\lambda;\varepsilon)\geq aE_0(\lambda;\varepsilon)\big\}\\ \geq \inf\left\{\frac{F\big(\sqrt{\lambda^2-\mu^2}\big)\sqrt{\lambda^2-\mu^2}}{\lambda}\mid \mu\in\sigma^0_{\mathrm{sym}}\cap[0,\lambda]\right\},\end{aligned}$$

where $E_0(\lambda;\varepsilon) := E_0((\lambda-\varepsilon,\lambda+\varepsilon))$.

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$$\geq \inf\left\{\frac{F\big(\sqrt{\lambda^2-\mu^2}\big)\sqrt{\lambda^2-\mu^2}}{\lambda} \mid \mu \in \sigma^0_{\mathrm{sym}} \cap [0,\lambda]\right\},$$

where $E_0(\lambda;\varepsilon) := E_0((\lambda-\varepsilon, \lambda+\varepsilon))$.

The right-hand side is strictly positive for all $\lambda \in (0,\infty) \setminus \sigma^0_{\mathrm{sym}}$.

7 Some references

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Thank you for your attention!