Time delay and Calabi invariant in classical scattering theory

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Starting observation

If $H_0(q,p) = |p|^2/2$ is the kinetic energy on $\mathbb{R}^{2n}$ and $\Phi_j(q,p) = q^j$ are the position observables, then

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• if $H \in C^\infty(\mathbb{R}^{2n})$ is a suitable perturbation of H_0, the perturbed trajectories corresponding to $\{\varphi_t^0(q,p)\}_{t\in\mathbb{R}}$ also escape from B_r

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What happens when H_0 and H are abstract Hamiltonians on a symplectic manifold M ?

Free Hamiltonian and position observables

(M, ω) is a symplectic manifold. For $f, g \in C^\infty(M)$, we define the Hamiltonian vector field X_f and the Poisson bracket $\{f, g\}$ by

$$\mathrm{d}f(\cdot) := \omega(X_f, \cdot) \quad \text{and} \quad \{f, g\} := \omega(X_f, X_g).$$

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$H_0 \in C^\infty(M)$ is an Hamiltonian with complete flow $\{\varphi_t^0\}_{t\in\mathbb{R}}$ and corresponding Hamiltonian evolution equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} f \circ \varphi_t^0 = \{f, H_0\} \circ \varphi_t^0, \quad t \in \mathbb{R}.$$

We consider a family of observables $\Phi \equiv (\Phi_1, \ldots, \Phi_d) \in C^\infty(M; \mathbb{R}^d)$ with

$$\partial_j H_0 := \{\Phi_j, H_0\} \quad \text{and} \quad \nabla H_0 := (\partial_1 H_0, \ldots, \partial_d H_0).$$

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The set of critical points

$$\mathsf{Crit}(H_0, \Phi) := (\nabla H_0)^{-1}(\{0\}) \subset M$$

is closed and contains the set $\mathsf{Crit}(H_0)$ of critical points of H_0:

$$\mathsf{Crit}(H_0, \Phi) \supset \mathsf{Crit}(H_0) \equiv \big\{m \in M \mid X_{H_0}(m) = 0\big\}.$$

Assumption 1.1 (Position observables)**.** *For each* $j = 1, 2, \ldots, d$*, we assume that* $\big\{\{\Phi_j, H_0\}, H_0\big\} = 0$*.*

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Thus, we have for $t \in \mathbb{R}$ and $m \in M$ that

$$\begin{aligned}\big(\Phi_j \circ \varphi_t^0\big)(m) &= \Phi_j(m) + t\,\{\Phi_j, H_0\}(m) + \frac{t^2}{2}\big\{\{\Phi_j, H_0\}, H_0\big\}(m) + \cdots \\ &= \Phi_j(m) + t\,\big(\partial_j H_0\big)(m),\end{aligned}$$

and each orbit $\{\varphi_t^0(m)\}_{t\in\mathbb{R}}$ stays in $\mathsf{Crit}(H_0, \Phi)$ if $m \in \mathsf{Crit}(H_0, \Phi)$, or stays outside $\mathsf{Crit}(H_0, \Phi)$ and is not periodic if $m \notin \mathsf{Crit}(H_0, \Phi)$.

Example 1.2 ($H_0(q,p) = h(p)$)**.** *$M := T^*\mathbb{R}^n \simeq \mathbb{R}^{2n}$, $\omega := \sum_{j=1}^n \mathrm{d}q^j \wedge \mathrm{d}p_j$, $H_0(q,p) := h(p)$ with $h \in C^\infty(\mathbb{R}^n;\mathbb{R})$, and $\Phi_j(q,p) := q^j$.*

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Then, $\varphi_t^0(q,p) = \big(q + t(\nabla h)(p), p\big)$, $\nabla H_0 = \nabla h$, *and*

$$\big\{\{\Phi_j, H_0\}, H_0\big\} = \big\{(\partial_j h)(p), h(p)\big\} = 0.$$

Furthermore, $\mathrm{Crit}(H_0) = \mathrm{Crit}(H_0, \Phi) = \mathbb{R}^n \times (\nabla h)^{-1}(\{0\})$.

Example 1.3 (Poincaré ball). *Let* $\mathring{B}_1 := \left\{q \in \mathbb{R}^n \mid |q| < 1\right\}$ *with*

$$g_q(X_q, Y_q) := \frac{4}{(1-|q|^2)^2}(X_q \cdot Y_q), \qquad X_q, Y_q \in T_q\mathring{B}_1 \simeq \mathbb{R}^n,$$

$$H_0 : T^*\mathring{B}_1 \to \mathbb{R}, \qquad (q,p) \mapsto \frac{1}{2}\sum_{j,k=1}^{n} g^{jk}(q)\, p_j\, p_k = \frac{1}{8}\,|p|^2\left(1-|q|^2\right)^2.$$

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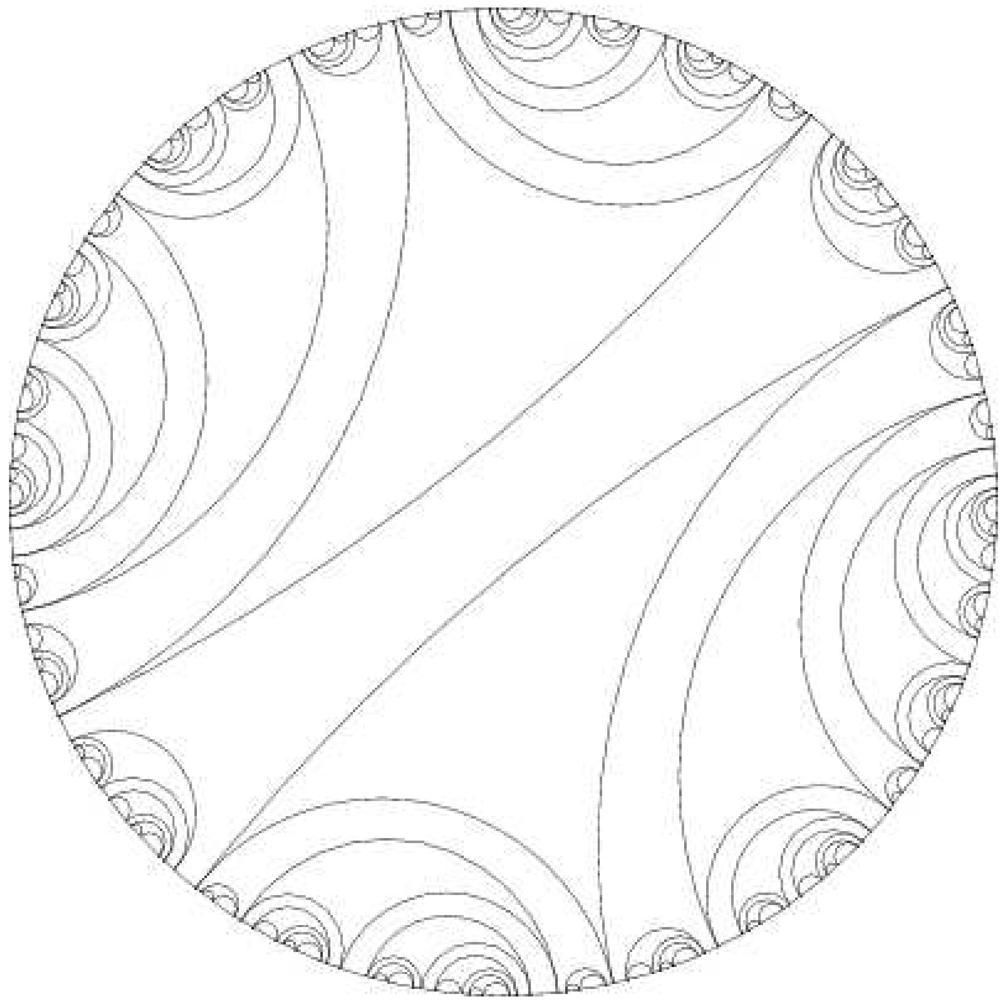
H_0 has complete flow on $M := T^*\mathring{B}_1 \setminus H_0^{-1}(\{0\}) \simeq \mathring{B}_1 \times \mathbb{R}^n \setminus \{0\}$,

$$\Phi : M \to \mathbb{R}, \quad (q,p) \mapsto \tanh^{-1}\left(\frac{2(p\cdot q)}{|p|(1+|q|^2)}\right),$$

satisfies Assumption 1.1 with $\nabla H_0 = \sqrt{2H_0}$, *and*
$\mathrm{Crit}(H_0) = \mathrm{Crit}(H_0, \Phi) = \varnothing$.

$\Phi(q,p)$ is the signed geodesic distance between q and the closest point to $0 \in \mathring{B}_1$ on the geodesic curve generated by (q,p).



Wave maps and scattering map

Assumption 1.4 (Potential). *$H \in C^\infty(M)$ has complete flow $\{\varphi_t\}_{t\in\mathbb{R}}$ and $V := H - H_0$ is of bounded support in Φ*

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The sets of Φ-bounded trajectories are

$$B_\Phi^\pm := \left\{m \in M \mid \exists\, R \geq 0 \text{ such that } \left|\Phi\big(\varphi_{\pm t}(m)\big)\right| \leq R \text{ for all } t \geq 0\right\}.$$

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Theorem 1.5 (Existence of wave maps). *Let H_0, H satisfy Assumptions 1.1 and 1.4. Then, the wave maps*

$$W_\pm := \lim_{t\to\pm\infty} \varphi_{-t} \circ \varphi_t^0$$

exist and are symplectomorphisms from $M \setminus \mathsf{Crit}(H_0, \Phi)$ to $M \setminus B_\Phi^\pm$.

Lemma 1.6 (Completeness of wave maps)**.** *Let* H_0, H *satisfy Assumptions 1.1 and 1.4, plus some technical condition. Assume there exists* $\delta > 0$ *such that* $\{\Phi \cdot \nabla H_0, H\}(m) > \delta$ *for all* $m \in M$*. Then,* $B_\Phi^\pm = \varnothing$ *and the wave maps*

$$W_\pm : M \setminus \mathsf{Crit}(H_0, \Phi) \to M$$

and the scattering map

$$S := W_+^{-1} \circ W_- : M \setminus \mathsf{Crit}(H_0, \Phi) \to M \setminus \mathsf{Crit}(H_0, \Phi)$$

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- The assumption $\{\Phi \cdot \nabla H_0, H\} > \delta$ is a virial-type condition coming from the requirement $\frac{\mathrm{d}^2}{\mathrm{d}t^2}\big(|\Phi|^2 \circ \varphi_t\big) > \delta$.

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- The assumption $\{\Phi \cdot \nabla H_0, H\} > \delta$ can be made local (no need for V to be globally "repulsive").

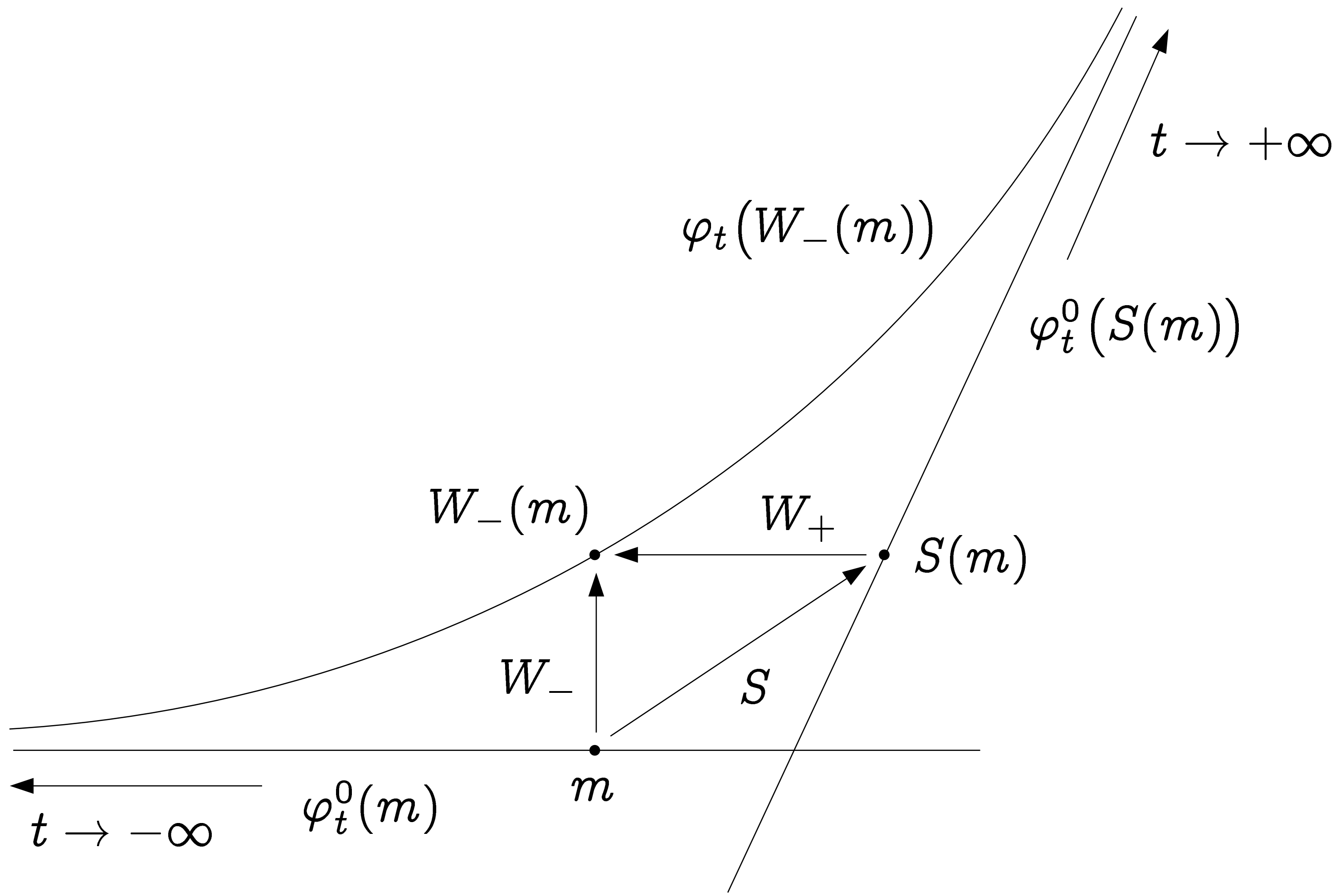

Figure 1: Wave maps $W_\pm$ and scattering map S

Time delay in classical scattering theory

Suppose for a moment that:

Assumption 1.7 (Wave maps)**.**

(i) $W_\pm = \lim_{t\to\pm\infty} \varphi_{-t} \circ \varphi_t^0$ *exist on some open sets* $\mathcal{D}_\pm \subset M$*.*

(ii) $W_\pm$ *are invertible, with* $W_\pm^{-1} : \mathsf{Ran}(W_\pm) \to \mathcal{D}_\pm$*.*

(iii) $W_\pm$ *have common ranges* $\mathsf{Ran}(W_+) = \mathsf{Ran}(W_-)$*.*

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(we have seen conditions guaranteeing this)

- χ_r^Φ, characteristic function for the set $\Phi^{-1}(B_r)$.

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- Sojourn time in $\Phi^{-1}(B_r)$ for the free trajectory starting from $m_- \in \mathcal{D}_-$ at $t = 0$:

$$T_r^0(m_-) := \int_{\mathbb{R}} \mathrm{d}t \left(\chi_r^\Phi \circ \varphi_t^0\right)(m_-).$$

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- Corresponding sojourn time for the perturbed trajectory starting from $W_-(m_-)$ at time $t = 0$:

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- Time delay in $\Phi^{-1}(B_r)$ for the scattering system (H_0, H) with starting point m_-:

$$\tau_r(m_-) := T_r(m_-) - \frac{1}{2}\big\{T_r^0(m_-) + \big(T_r^0 \circ S\big)(m_-)\big\}.$$

Theorem 1.8 (Time delay). *Let H_0 and H satisfy Assumptions 1.1 and 1.7, and let $m_- \in \mathcal{D}_- \setminus \mathsf{Crit}(H_0, \Phi)$ satisfy $S(m_-) \notin \mathsf{Crit}(H_0, \Phi)$, plus some technical condition. Then,*

$$\lim_{r\to\infty} \tau_r(m_-) = T(m_-) - (T \circ S)(m_-)$$

with $T : M \setminus \mathsf{Crit}(H_0, \Phi) \to \mathbb{R}$ the C^∞-function given by

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$\big(-T(m)$ is the (arrival) time at which a particle in $\mathbb{R}^d$ with initial position $\Phi(m)$ and velocity $(\nabla H_0)(m)$ intersects the hyperplane orthogonal to the unit vector $\frac{(\nabla H_0)(m)}{|(\nabla H_0)(m)|}\big)$

Some comments:

- Set $\tau(m_-) := \lim_{r\to\infty} \tau_r(m_-)$. Since

$$T \circ \varphi_t^0 = T + t \quad \text{and} \quad \varphi_t^0 \circ S = S \circ \varphi_t^0,$$

one has

$$\big(\tau \circ \varphi_t^0\big)(m_-) = \big\{(T - T \circ S) \circ \varphi_t^0\big\}(m_-) = \tau(m_-),$$

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• The formula of the theorem should be compared to the Eisenbud-Wigner formula of quantum mechanics:

$$\begin{aligned}\lim_{r\to\infty} \tau_r(\varphi) = \langle \varphi, T\varphi\rangle_{\mathcal{H}} - \langle S\varphi, TS\varphi\rangle_{\mathcal{H}} &= -\langle \varphi, S^*[T,S]\varphi\rangle_{\mathcal{H}} \\ &= -\Big\langle \varphi, iS^* \frac{\mathrm{d}S}{\mathrm{d}H_0}\,\varphi\Big\rangle_{\mathcal{H}}.\end{aligned}$$

- If

$$|(\nabla H_0)(m_-)|^2 = |(\nabla H_0 \circ S)(m_-)|^2,$$

then one can replace in the theorem the symmetrized time delay $\tau_r(m_-)$ by the unsymmetrized time delay

$$\tau_r^{\text{in}}(m_-) := T_r(m_-) - T_r^0(m_-).$$

Example 1.9 ($H_0(q,p) = h(p)$, continued)**.** *Let* $H(q,p) := h(p) + V(q)$ *with* $V \in C_c^\infty(\mathbb{R}^n;\mathbb{R})$*. Under some conditions on* h*, one can find an open subset* $U \subset \mathbb{R}^{2n}$ *on which all the assumptions are satisfied.*

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So, the theorem on time delay applies, and one has for $(q_-, p^-) \in U$

$$\begin{aligned}\lim_{r\to\infty} \tau_r(q_-, p^-) &= T(q_-, p^-) - T(q_+, p^+) \\ &= \frac{q_- \cdot (\nabla h)(p^-)}{|(\nabla h)(p^-)|^2} - \frac{q_+ \cdot (\nabla h)(p^+)}{|(\nabla h)(p^+)|^2},\end{aligned}$$

with $(q_+, p^+) := S(q_-, p^-)$.

Example 1.10 (Poincaré ball, continued)**.** *Let* $H(q,p) := H_0(q,p) + V(q)$ *with* $V \in C^\infty(\mathring{B}_1;\mathbb{R})$*. One can find an open subset* $U \subset M$ *on which all the assumptions are satisfied.*

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$$= \lim_{r\to\infty} \tau_r(q_-, p^-) = T(q_-, p^-) - T(q_+, p^+)$$

$$\vdots$$

$$= \frac{2\left\{\tanh^{-1}\left(\frac{2(p^-\cdot q_-)}{|p^-|(1+|q_-|^2)}\right) - \tanh^{-1}\left(\frac{2(p^+\cdot q_+)}{|p^+|(1+|q_+|^2)}\right)\right\}}{|p^-|(1-|q_-|^2)},$$

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Calabi invariant of the Poincaré scattering map

Let (M', ω') be an exact $2n'$-dimensional symplectic manifold with $\omega' = \mathrm{d}\alpha$ for some 1-form α. Let ψ be a symplectomorphism of M' with compact support such that

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- $\mathrm{Cal}(\psi)$ is independent of the choice of α,
- the restriction $\mathrm{Cal}|_{\{\text{hamiltomorphisms with compact support}\}}$ is a homomorphism (see *e.g.* [McDuff/Salamon 98]).

Assume that M is exact with $\dim(M) \geq 4$. Suppose that

- Assumption 1.1 holds,
- V has compact support (plus some technical condition),
- $U \subset \mathbb{R}$ is an open set such that
  (i) $H_0^{-1}(U) \cap \mathsf{Crit}(H_0, \Phi) = \varnothing$,
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Then

- The maps

$$W_\pm : H_0^{-1}(U) \to H^{-1}(U) \quad \text{and} \quad S : H_0^{-1}(U) \to H_0^{-1}(U)$$

  are well defined symplectomorphisms.

- For each $E \in U$, $\Sigma_E^0 := H_0^{-1}(\{E\})$ is regular submanifold of M and

$$\Gamma_E = \{m \in \Sigma_E^0 \mid (\Phi \cdot \nabla H_0)(m) = 0\}.$$

is (in Σ_E^0) a local transversal section of the vector field $X_{H_0}|_{\Sigma_E^0}$ (see [Abraham/Marsden 78]).

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- The orbit space $\widetilde{\Sigma}_E^0 := \Sigma_E^0/\mathbb{R}$, *i.e.* the quotient of Σ_E^0 by the group action

$$\varphi^{0,E} : \mathbb{R} \times \Sigma_E^0 \to \Sigma_E^0, \quad (t, m) \mapsto \varphi_t^{0,E}(m),$$

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- The restricted scattering map $S_E := S|_{\Sigma_E^0}$ induces a symplectomorphism $\widetilde{S}_E$ on $\widetilde{\Sigma}_E^0$ (called the Poincaré scattering map) with compact support.

Using a theorem of [Buslaev/Pushnitski 10] giving the expression of $\frac{\mathrm{d}}{\mathrm{d}E}\mathrm{Cal}(\widetilde{S}_E)$ in terms of integrals over transversal sections, we obtain :

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Theorem 1.11 (Calabi invariant). *Under the preceding assumptions, one has*

$$\frac{\mathrm{d}}{\mathrm{d}E}\,\mathrm{Cal}\big(\widetilde{S}_E\big) = -\int_{\Gamma_E} \lim_{r\to\infty} \tau_r(m)\,\frac{\omega^{n-1}(m)}{(n-1)\,!}$$
$$= \int_{\Gamma_E} \frac{(\Phi\circ S)(m)\cdot(\nabla H_0\circ S)(m)}{|(\nabla H_0\circ S)(m)|^2}\,\frac{\omega^{n-1}(m)}{(n-1)\,!}\,.$$

Example 1.12 ($H_0(q,p) = h(p)$, the end)**.** *If the dimension of $M \simeq \mathbb{R}^{2n}$ is ≥ 4 and V has compact support, then all the assumptions are verified for an open set $U \subset \mathbb{R}$.*

Example 1.12 ($H_0(q,p)=h(p)$, the end)**.** *If the dimension of* $M\simeq\mathbb{R}^{2n}$ *is* ≥ 4 *and* V *has compact support, then all the assumptions are verified for an open set* $U\subset\mathbb{R}$.

So, the theorem on the Calabi invariant applies, and one has for each $E\in U$

$$\frac{\mathrm{d}}{\mathrm{d}E}\,\mathrm{Cal}\big(\widetilde{S}_E\big)=\int_{\{(q,p)\in\mathbb{R}^{2n}\,|\,h(p)=E,\ q\cdot(\nabla h)(p)=0\}}\frac{q_+\cdot(\nabla h)(p^+)}{|(\nabla h)(p^+)|^2}\,\frac{\omega^{n-1}(q,p)}{(n-1)!},$$

with $(q_+,p^+):=S(q,p)$.

Example 1.12 ($H_0(q,p) = h(p)$, the end)**.** *If the dimension of $M \simeq \mathbb{R}^{2n}$ is ≥ 4 and V has compact support, then all the assumptions are verified for an open set $U \subset \mathbb{R}$.*

So, the theorem on the Calabi invariant applies, and one has for each $E \in U$

$$\frac{\mathrm{d}}{\mathrm{d}E}\,\mathrm{Cal}\big(\widetilde{S}_E\big) = \int_{\{(q,p)\in\mathbb{R}^{2n}\,|\,h(p)=E,\ q\cdot(\nabla h)(p)=0\}} \frac{q_+\cdot(\nabla h)(p^+)}{|(\nabla h)(p^+)|^2}\,\frac{\omega^{n-1}(q,p)}{(n-1)!}\,,$$

with $(q_+, p^+) := S(q,p)$.

In the case $h(p) = |p|^2/2$, one obtains

$$\frac{\mathrm{d}}{\mathrm{d}E}\,\mathrm{Cal}\big(\widetilde{S}_E\big) = (2E)^{(n-3)/2}\int_{\mathbb{S}^{n-1}}\mathrm{d}^{n-1}\widehat{p}\int_{q\cdot p=0}\mathrm{d}^{n-1}q\,\big(q_+\cdot p^+\big),$$

with $\widehat{p} := p/|p|$ and $\mathrm{d}^{n-1}\widehat{p}$ the spherical measure on $\mathbb{S}^{n-1}$.

Example 1.13 (Poincaré ball, the end). *If the dimension of* $M \simeq \mathring{B}_1 \times \mathbb{R}^n \setminus \{0\}$ *is* ≥ 4 *and* V *has compact support, then all the assumptions are verified for an open set* $U \subset \mathbb{R}$.

Example 1.13 (Poincaré ball, the end). *If the dimension of $M \simeq \mathring{B}_1 \times \mathbb{R}^n \setminus \{0\}$ is ≥ 4 and V has compact support, then all the assumptions are verified for an open set $U \subset \mathbb{R}$.*

So, the theorem on the Calabi invariant applies, and one has for each $E \in U$

$$\frac{\mathrm{d}}{\mathrm{d}E}\,\mathrm{Cal}\big(\widetilde{S}_E\big)$$
$$= (2E)^{-1/2} \int_{\{(q,p)\in\mathring{B}_1\times\mathbb{R}^n\setminus\{0\} \mid |p|^2(1-|q|^2)^2=8E,\ p\cdot q=0\}} \Phi(q_+,p^+)\,\frac{\omega^{n-1}(q,p)}{(n-1)!}\,,$$

with $\Phi(q,p) = \tanh^{-1}\left(\frac{2(p\cdot q)}{|p|(1+|q|^2)}\right)$ and $(q_+,p^+) := S(q,p)$.

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