Decay estimates for unitary representations with applications to continuous- and discrete-time models

Rafael Tiedra de Aldecoa

Pontifical Catholic University of Chile

Torun, October 2021

Joint work with Serge Richard (Nagoya)

Table of Contents

General case

- $\mathcal{H}$, Hilbert space with norm $\|\cdot\|_{\mathcal{H}}$ and scalar product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$
- $\mathscr{B}(\mathcal{H})$, set of bounded linear operators on $\mathcal{H}$
- $\mathscr{K}(\mathcal{H})$, set of compact operators on $\mathcal{H}$
- A, self-adjoint operator in $\mathcal{H}$ with domain $\mathcal{D}(A)$

Definition

$S \in C^k(A)$ if $S \in \mathscr{B}(\mathcal{H})$ and if the map

$$\mathbb{R} \ni t \mapsto \mathrm{e}^{-itA} S \, \mathrm{e}^{itA} \in \mathscr{B}(\mathcal{H})$$

is strongly of class C^k.

Intuitively, if $S \in C^k(A)$, then the k-th commutator

$$\big[\cdots \big[[S, \underbrace{A], A\big], \dots, A\big]}_{k \text{ times}}$$

is a well-defined bounded operator.

Theorem

Let $(U_j)_{j\in J}$ *be a net in* $\mathrm{U}(\mathcal{H})$*, let* $(\ell_j)_{j\in J}\subset[0,\infty)$ *satisfy* $\ell_j\to\infty$*, let* A *be self-adjoint in* $\mathcal{H}$ *with* $U_j\in C^1(A)$ *for each* $j\in J$*, and suppose that*

$$D:=\operatorname*{s-lim}_j D_j \quad \textit{with} \quad D_j:=\tfrac{1}{\ell_j}[A,U_j]U_j^{-1}$$

exists.

(a) *For each* $\varphi=D\widetilde{\varphi}\in D\mathcal{D}(A)$ *and* $\psi\in\mathcal{D}(A)$ *there is* $c_{\varphi,\psi}\geq 0$ *such that*

$$|\langle\varphi,U_j\psi\rangle_{\mathcal{H}}|\leq\|(D-D_j)\widetilde{\varphi}\|_{\mathcal{H}}\,\|\psi\|_{\mathcal{H}}+\tfrac{1}{\ell_j}\,c_{\varphi,\psi},\quad \ell_j>0.$$

In particular, $\lim_j\langle\xi,U_j\zeta\rangle_{\mathcal{H}}=0$ *for all* $\xi\in\ker(D)^\perp$ *and* $\zeta\in\mathcal{H}$*.*

Theorem (Continued)

(b) *Assume that* $D = D_j$ *for all* $j \in J$. *Then for each* $\varphi \in D\mathcal{D}(A)$ *and* $\psi \in \mathcal{D}(A)$ *there is* $c_{\varphi,\psi} \geq 0$ *such that*

$$|\langle \varphi, U_j \psi \rangle_{\mathcal{H}}| \leq \tfrac{1}{\ell_j}\, c_{\varphi,\psi}, \quad \ell_j > 0.$$

(c) *Assume that* $D = D_j$ *for all* $j \in J$, *that* $D \in C^1(A)$, *and that* $[A, D] = DB$ *with* $B \in C^{(n-1)}(A)$ *and* $[D, B] = 0$ $(n \geq 1)$. *Then for each* $\varphi \in D^n\mathcal{D}(A^n)$ *and* $\psi \in \mathcal{D}(A^n)$ *there is* $c_{\varphi,\psi} \geq 0$ *such that*

$$|\langle \varphi, U_j \psi \rangle_{\mathcal{H}}| \leq \tfrac{1}{\ell_j^n}\, c_{\varphi,\psi}, \quad \ell_j > 0.$$

Remark

If $[A, \cdot]$ is seen as a derivation on the set (U_j), then D corresponds to an operator-valued winding number for the map $j \mapsto U_j$ (the usual logarithmic derivative $\frac{\mathrm{d}z}{z}$ is replaced by the "logaritmic derivative" $[A, U_j]U_j^{-1}$).

Remark

The U_j can be given by a representation $\mathscr{U} : X \to \mathrm{U}(\mathcal{H})$ of a topological group X and the ℓ_j by a proper length function $\ell : X \to [0, \infty)$.

Remark

If the U_j are given by a representation, then the property

$$\lim_j \langle \xi, U_j \zeta \rangle_{\mathcal{H}} = 0, \quad \xi \in \ker(D)^{\perp},\ \zeta \in \mathcal{H},$$

is a mixing property of the representation in $\ker(D)^{\perp}$.

Idea of the proof.

(a)

$$
\begin{aligned}
|\langle \varphi, U_j\psi\rangle_{\mathcal{H}}| &= \left|\langle (D-D_j)\widetilde{\varphi}, U_j\psi\rangle_{\mathcal{H}} + \langle D_j\widetilde{\varphi}, U_j\psi\rangle_{\mathcal{H}}\right| \\
&\le \|(D-D_j)\widetilde{\varphi}\|_{\mathcal{H}}\,\|\psi\|_{\mathcal{H}} + \tfrac{1}{\ell_j}\left|\langle [A,U_j]U_j^{-1}\widetilde{\varphi}, U_j\psi\rangle_{\mathcal{H}}\right| \\
&\le \|(D-D_j)\widetilde{\varphi}\|_{\mathcal{H}}\,\|\psi\|_{\mathcal{H}} + \tfrac{1}{\ell_j}\left|\langle A\widetilde{\varphi}, U_j\psi\rangle_{\mathcal{H}}\right| + \tfrac{1}{\ell_j}\left|\langle \widetilde{\varphi}, U_jA\psi\rangle_{\mathcal{H}}\right| \\
&\le \|(D-D_j)\widetilde{\varphi}\|_{\mathcal{H}}\,\|\psi\|_{\mathcal{H}} + \tfrac{1}{\ell_j}\,c_{\varphi,\psi}
\end{aligned}
$$

with $c_{\varphi,\psi} := \|A\widetilde{\varphi}\|_{\mathcal{H}}\|\psi\|_{\mathcal{H}} + \|\widetilde{\varphi}\|_{\mathcal{H}}\|A\psi\|_{\mathcal{H}}$.

(b) Direct consequence of (a).

(c) Induction over n, starting from (b) for $n=1$. □

Unitary representations with self-adjoint generator

Suppose that the U_j are given by a strongly continuous representation $\mathscr{U} : \mathbb{R} \to \mathsf{U}(\mathcal{H})$.

Then Stone's theorem implies the existence of a self-adjoint operator H such that $\mathscr{U}(t) = \mathrm{e}^{-itH}$ for each $t \in \mathbb{R}$.

— continuous time case —

Proposition (The case $[iH, A] = f(H)$)

Let H and A be self-adjoint in $\mathcal{H}$, assume that $(H - i)^{-1} \in C^1(A)$ with $[iH, A] = f(H)$ for some $f : \mathbb{R} \to \mathbb{R}$, and set $g(x) := f(x)(1 + x^2)^{-1}$.

(a) *For each $\varphi \in g(H)\mathcal{D}(A)$ and $\psi \in \mathcal{D}(A)$ there is $c_{\varphi,\psi} \geq 0$ such that*

$$\left|\langle \varphi, \mathrm{e}^{-itH}\, \psi \rangle_{\mathcal{H}}\right| \leq \tfrac{1}{t}\, c_{\varphi,\psi}, \quad t > 0.$$

(b) *If $g(H)\mathcal{D}(A) \subset \mathcal{D}(A)$, then $H|_{\ker(f(H))^\perp}$ has purely a.c. spectrum.*

(c) *Suppose that $f \in C^n(\mathbb{R})$ $(n \geq 1)$ with $g^{(k)} \in \mathrm{L}^2(\mathbb{R}) \cap \mathrm{L}^\infty(\mathbb{R})$ for $k = 0, \ldots, n$ and $g^{(n)}$ uniformly continuous. Then for each $\varphi \in g(H)^n\mathcal{D}(A^n)$ and $\psi \in \mathcal{D}(A^n)$ there is $c_{\varphi,\psi} \geq 0$ such that*

$$\left|\langle \varphi, \mathrm{e}^{-itH}\, \psi \rangle_{\mathcal{H}}\right| \leq \tfrac{1}{t^n}\, c_{\varphi,\psi}, \quad t > 0.$$

Idea of the proof.

(a) Apply point (b) of the theorem in the case $(U_j)_{j\in J} = (\mathrm{e}^{-itH})_{t>0}$, the set $(\ell_j)_{j\in J} = (t)_{t>0}$, the operator $\widetilde{A} = (H+i)^{-1}A(H-i)^{-1}$, and

$$\begin{aligned} D_t = \tfrac{1}{t}[\widetilde{A}, \mathrm{e}^{-itH}]\,\mathrm{e}^{itH} &= \tfrac{1}{t}\int_0^t \mathrm{d}\tau\ \mathrm{e}^{-i\tau H}(H+i)^{-1}[iH, A](H-i)^{-1}\,\mathrm{e}^{i\tau H} \\ &= g(H) \\ &= D. \end{aligned}$$

(b) Point (a) implies that $t \mapsto \langle \psi, \mathrm{e}^{-itH}\,\psi\rangle_{\mathcal{H}}$ is in $\mathrm{L}^2(\mathbb{R})$ for suitable $\psi \in \mathcal{H}$. Then Plancherel's theorem implies that the spectral measures $m_\psi(\cdot) := \|E^H(\cdot)\psi\|^2_{\mathcal{H}}$ are a.c.

(c) The assumptions on f and g guarantee that point (c) of the theorem applies. □

Unitary representations with unitary generator

Suppose that the U_j are given by a representation $\mathscr{U} : \mathbb{Z} \to \mathrm{U}(\mathcal{H})$.

Then, since $\mathbb{Z}$ has generator 1, there exists a unitary operator U such that $\mathscr{U}(m) = U^m$ for each $m \in \mathbb{Z}$.

— discrete time case —

Proposition (The case $[A, U] = \gamma(U)$)

Let U and A be unitary and self-adjoint in $\mathcal{H}$, assume that $U \in C^1(A)$ with $[A, U] = \gamma(U)$ for some $\gamma : \mathbb{S}^1 \to \mathbb{C}$, and set $\eta(U) := \gamma(U)U^{-1}$.

(a) *For each $\varphi \in \eta(U)\mathcal{D}(A)$ and $\psi \in \mathcal{D}(A)$ there is $c_{\varphi,\psi} \geq 0$ such that*

$$\big|\langle\varphi, U^n\psi\rangle_{\mathcal{H}}\big| \leq \tfrac{1}{n}\, c_{\varphi,\psi}, \quad n \geq 1.$$

(b) *If $\eta(U)\mathcal{D}(A) \subset \mathcal{D}(A)$, then $U|_{\ker(\gamma(U))^\perp}$ has purely a.c. spectrum.*

(c) *Suppose that $\gamma \in C^k(\mathbb{S}^1)$ ($k \geq 1$). Then for each $\varphi \in \eta(U)^k\mathcal{D}(A^k)$ and $\psi \in \mathcal{D}(A^k)$ there is $c_{\varphi,\psi} \geq 0$ such that*

$$\big|\langle\varphi, U^n\psi\rangle_{\mathcal{H}}\big| \leq \tfrac{1}{n^k}\, c_{\varphi,\psi}, \quad n \geq 1.$$

Idea of the proof.

Similar to the self-adjoint case. □

Examples

The theory applies to various models:

- Left regular representation
- Schrödinger operator in $\mathbb{R}^n$
- Dirac operator in $\mathbb{R}^3$
- Quantum waveguides in $\mathbb{R}^n$
- Stark Hamiltonian in $\mathbb{R}^n$
- Fractional Laplacian in $\mathbb{R}^n$
- Horocycle flow

- Adjacency matrices
- Jacobi matrices
- Schrödinger operators on Fock spaces
- Multiplication by λ in $\mathsf{L}^2(\mathbb{R}_+, \mathsf{d}\mu)$
- $H = -\partial_{xx} + \partial_{yy}$ in $\mathbb{R}^2$
- $H = -X^{2-s}\Delta - \Delta X^{2-s}$ in $\mathbb{R}_+$
- Quantum walks on $\mathbb{Z}$
- Quantum walks on trees
- Skew products

Left regular representation

- X, σ-compact locally compact Hausdorff group with identity e and left Haar measure μ.
- ℓ, proper length function on X,
  (L1) $\ell(e) = 0$,
  (L2) $\ell(x^{-1}) = \ell(x)$ for all $x \in X$,
  (L3) $\ell(xy) \leq \ell(x) + \ell(y)$ for all $x, y \in X$,
  (L4) if $K \subset [0, \infty)$ is compact, then $\ell^{-1}(K) \subset X$ is relatively compact.
- $\mathscr{U} : X \to \mathrm{U}(\mathcal{H})$, left regular representation of X on $\mathcal{H}$

$$\mathscr{U}(x)\varphi := \varphi(x^{-1}\cdot), \quad x \in X,\ \varphi \in \mathcal{H} := \mathrm{L}^2(X, \mu).$$

Let A be the operator of multiplication by ℓ

$$A\varphi := \ell\varphi, \quad \varphi \in \mathcal{D}(A) := \{\varphi \in \mathcal{H} \mid \|\ell\varphi\|_{\mathcal{H}} < \infty\}.$$

Then for any net $(x_j)_{j\in J} \subset X$ with $x_j \to \infty$ we have

$$D := \operatorname*{s-lim}_j D_j = \operatorname*{s-lim}_j \tfrac{1}{\ell(x_j)}[A, \mathscr{U}(x_j)]\mathscr{U}(x_j)^{-1} = \cdots = -1.$$

Thus point (a) of the theorem implies that for each $\varphi, \psi \in \mathcal{D}(A)$ there is $c_{\varphi,\psi} \geq 0$ such that

$$\left|\langle\varphi, \mathscr{U}(x_j)\psi\rangle_{\mathcal{H}}\right| \leq \left\|(D - D_j)\varphi\right\|_{\mathcal{H}} \|\psi\|_{\mathcal{H}} + \tfrac{1}{\ell(x_j)}\, c_{\varphi,\psi}, \quad \ell(x_j) > 0.$$

(new proof - not using convolutions - of the mixing property of the left regular representation)

Remark

Doing explicit computations, one can show higher decay estimates such as

$$\left|\langle\varphi, \mathscr{U}(x_j)\psi\rangle_{\mathcal{H}}\right| \leq \frac{1}{\ell(x_j)}\, c_{\varphi,\psi} \quad \textit{or} \quad \left|\langle\varphi, \mathscr{U}(x_j)\psi\rangle_{\mathcal{H}}\right| \leq \frac{1}{\ell(x_j)^2}\, c_{\varphi,\psi}.$$

Remark

In general $\mathscr{U}$ doesn't have neither a self-adjoint generator nor a unitary generator.

Horocycle flow

- Σ, finite volume Riemann surface of genus ≥ 2
- $M := T^1\Sigma$, unit tangent bundle of Σ with probability measure μ_Ω induced by volume form Ω
- μ_Ω-preserving flows: horocycle $F_1 := (F_{1,t})_{t\in\mathbb{R}}$ and geodesic $F_2 := (F_{2,t})_{t\in\mathbb{R}}$
- Self-adjoint generator H_j in $\mathcal{H} := \mathrm{L}^2(M, \mu_\Omega)$,

$$H_j\varphi := -i\mathcal{L}_{X_j}\varphi, \quad \varphi \in C_{\mathrm{c}}^\infty(M),$$

with X_j the vector field associated to F_j and $\mathcal{L}_{X_j}$ its Lie derivative

One has $(H_1 - i)^{-1} \in C^1(H_2)$ with $[iH_1, H_2] = H_1$. So, the proposition applies with $f(H_1) = H_1$, $g(H_1) = H_1(1 + H_1^2)^{-1}$ and $\ker(f(H_1)) = \ker(H_1)$.

Thus $H_1|_{\ker(H_1)^\perp}$ has purely a.c. spectrum (well-known) and for each $\varphi \in g(H_1)^n \mathcal{D}(H_2^n)$ and $\psi \in \mathcal{D}(H_2^n)$ $(n \geq 1)$ there is $c_{\varphi,\psi} \geq 0$ such that

$$\big|\langle \varphi, \psi \circ F_{1,t} \rangle_{\mathcal{H}}\big| = \big|\langle \varphi, \mathrm{e}^{-itH_1}\, \psi \rangle_{\mathcal{H}}\big| \leq \tfrac{1}{t^n}\, c_{\varphi,\psi}, \quad t > 0.$$

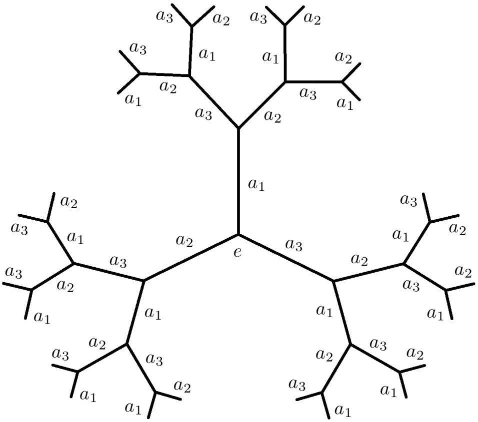
(new proof of polynomial decay of correlations for the horocycle flow not using the identification $M \simeq \Gamma \setminus \mathrm{PSL}(2, \mathbb{R})$ or representation theory)

Remark

In the case of a time-change of the horocycle flow and Σ compact, one "only" obtains the mixing property with this approach.

Quantum walks on trees

- $\mathcal{T} := \langle a_1, \ldots, a_d \mid a_1^2 = \cdots = a_d^2 = e \rangle$, homogeneous tree of odd degree $d \geq 3$ and word length $|\cdot|$



- $\mathcal{T}_{\mathrm{e}} := \{x \in \mathcal{T} \mid |x| \in 2\mathbb{N}\}$ and $\mathcal{T}_{\mathrm{o}} := \{x \in \mathcal{T} \mid |x| \in 2\mathbb{N}+1\}$ with characteristic functions $\chi_{\mathrm{e}} := \chi_{\mathcal{T}_{\mathrm{e}}}$ and $\chi_{\mathrm{o}} := \chi_{\mathcal{T}_{\mathrm{o}}}$.
- Quantum walk with evolution operator $U := SC$ in $\mathcal{H} := \ell^2(\mathcal{T}, \mathbb{C}^d)$

$$S := \begin{pmatrix} S_{1+1,1+2} & & 0 \\ & \ddots & \\ 0 & & S_{d+1,d+2} \end{pmatrix}, \quad S_{d,d+1} := S_{d,1},\ S_{d+1,d+2} := S_{1,2},$$

$$S_{i,j} f := \chi_{\mathrm{e}} f(\,\cdot\, a_i) + \chi_{\mathrm{o}} f(\,\cdot\, a_j), \quad i,j \in \{1,\ldots,d\},\ f \in \ell^2(\mathcal{T}),$$

$$(C\varphi)(x) := C(x)\varphi(x), \quad \varphi \in \mathcal{H},\ x \in \mathcal{T},\ C(x) \in \mathrm{U}(d).$$

Assumption (Short-range)

For $i = 1,\ldots,d$, there is a diagonal matrix $C_i \in \mathrm{U}(d)$ and $\varepsilon_i > 0$ such that

$$\big\| C(x) - C_i \big\|_{\mathscr{B}(\mathbb{C}^d)} \le \mathrm{Const.}\,\big(1 + |x|^2\big)^{-(1+\varepsilon_i)/2} \quad \text{if } x \in \mathcal{T}_i$$

where $\mathcal{T}_i := \big\{x \in \mathcal{T} \mid \text{the first letter of } x \in \mathcal{T} \text{ is } a_i\big\}$.

There exist A self-adjoint in $\mathcal{H}$, U_0 unitary in $\mathcal{H}_0$, A_0 self-adjoint in $\mathcal{H}_0$ and $J \in \mathscr{B}(\mathcal{H}_0, \mathcal{H})$ such that $U \in C^1(A)$, $U_0 \in C^\infty(A_0)$ and

$$JJ^* = 1_{\mathcal{H}}, \quad [A_0, U_0]U_0^{-1} = 2, \quad [A, U]U^{-1} - J[A_0, U_0]U_0^{-1}J^* \in \mathscr{K}(\mathcal{H}).$$

Then, by compacity arguments:

$$\begin{aligned} D = \operatorname*{s-lim}_{n\to\infty} D_n &= \operatorname*{s-lim}_{n\to\infty} \tfrac{1}{n}[A, U^n]U^{-n} \\ &= \operatorname*{s-lim}_{n\to\infty} \tfrac{1}{n} \sum_{m=0}^{n-1} U^m([A, U]U^{-1})U^{-m} \\ &= \operatorname*{s-lim}_{n\to\infty} \tfrac{1}{n} \sum_{m=0}^{n-1} U^m(J[A_0, U_0]U_0^{-1}J^*)U^{-m}P_{\rm c}(U) \qquad (\,!\,) \\ &= 2P_{\rm c}(U) \end{aligned}$$

with $P_{\rm c}(U)$ the projection onto the continuous subspace of U.

Thus point (a) of the theorem implies that for each $\varphi = D\widetilde{\varphi} \in D\mathcal{D}(A)$ and $\psi \in \mathcal{D}(A)$ there is $c_{\varphi,\psi} \geq 0$ such that

$$\big|\langle \varphi, U^n \psi \rangle_{\mathcal{H}}\big| \leq \|(D - D_n)\widetilde{\varphi}\|_{\mathcal{H}} \|\psi\|_{\mathcal{H}} + \tfrac{1}{n} c_{\varphi,\psi}, \quad n \geq 1.$$

Remark

One cannot easily improve this estimate, because its proof uses RAGE theorem, whose proof relies on a discrete version of Wiener's theorem, which doesn't come with an explicit rate of convergence.

Gracias !

References

- V. Georgescu, M. Larenas and A. Soffer. Abstract theory of pointwise decay with applications to wave and Schrödinger equations. *Ann. Henri Poincaré*, 2016
- S. Richard and R. Tiedra de Aldecoa. Commutator criteria for strong mixing II. More general and simpler. *Cubo*, 2019
- S. Richard and R. Tiedra de Aldecoa. Decay estimates for unitary representations with applications to continuous- and discrete-time models. Preprint on ArXiv, 2021
- R. Tiedra de Aldecoa. Commutator criteria for strong mixing. *Ergodic Theory Dynam. Systems*, 2017