Degree, mixing, and absolutely continuous spectrum of cocycles with values in compact Lie groups

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- $\{F_t\}_{t\in\mathbb{R}}$, C^1 measure-preserving flow on X with Lie derivative $\mathscr{L}_Y$
- G, compact Lie group with Haar measure μ_G, identity e_G, Lie algebra $\mathfrak{g}$, and Lie bracket $[\cdot,\cdot]_{\mathfrak{g}} : \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$

A measurable function $\phi : X \to G$ induces a measurable cocycle over F_1

$$X \times \mathbb{Z} \ni (x, n) \mapsto \phi^{(n)}(x) \in G$$

given by

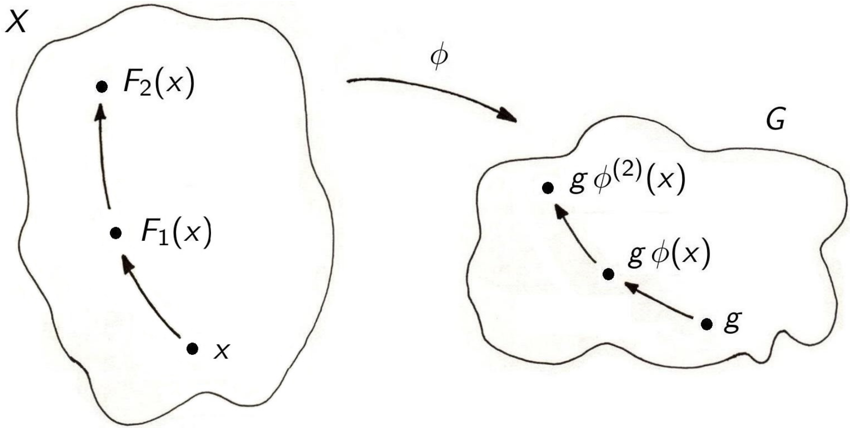
$$\phi^{(n)}(x) := \begin{cases} \phi(x)(\phi\circ F_1)(x)\cdots(\phi\circ F_{n-1})(x) & \text{if } n \geq 1 \\ e_G & \text{if } n = 0 \\ \left(\phi^{(-n)}\circ F_n\right)(x)^{-1} & \text{if } n \leq -1. \end{cases}$$

The skew product associated to ϕ is the measure preserving map

$$T_\phi : X \times G \to X \times G, \quad (x, g) \mapsto \big(F_1(x), g\,\phi(x)\big),$$

with iterates

$$T_\phi^n(x, g) = \big(F_n(x), g\,\phi^{(n)}(x)\big), \quad n \in \mathbb{Z}.$$



The Koopman operator for T_ϕ is the unitary operator

$$U_\phi \psi := \psi \circ T_\phi, \quad \psi \in \mathcal{H} := \mathrm{L}^2(X \times G, \mu_X \otimes \mu_G).$$

Peter-Weyl's theorem gives an orthogonal decomposition

$$\mathcal{H} = \bigoplus_{\pi \in \widehat{G}} \bigoplus_{j=1}^{d_\pi} \mathcal{H}_j^{(\pi)}, \quad \mathcal{H}_j^{(\pi)} := \bigoplus_{k=1}^{d_\pi} \mathrm{L}^2(X, \mu_X) \otimes \mathbb{C}\pi_{jk},$$

with $\pi : G \to \mathrm{U}(d_\pi)$ the finite-dimensional irreducible unitary representations of G, and $U_{\phi,\pi,j} := U_\phi\big|_{\mathcal{H}_j^{(\pi)}}$ the restriction given by

$$U_{\phi,\pi,j} \sum_{k=1}^{d_\pi} \varphi_k \otimes \pi_{jk} = \sum_{k,\ell=1}^{d_\pi} \big(\varphi_k \circ F_1\big)\big(\pi_{\ell k} \circ \phi\big) \otimes \pi_{j\ell}, \quad \varphi_k \in \mathrm{L}^2(X, \mu_X).$$

Degree of a cocycle

Definition (Degree of ϕ)

Assume that $\phi \in C^1(X, G)$ and let $M_\phi := \mathscr{L}_Y \phi \cdot \phi^{-1} \in C(X, \mathfrak{g})$. Then, the degree of ϕ is the function $P_\phi M_\phi : X \to \mathfrak{g}$ given for μ_X-almost every $x \in X$ by

$$\begin{aligned}(P_\phi M_\phi)(x) &:= \lim_{N\to\infty} \frac{1}{N} \big(\mathscr{L}_Y \phi^{(N)}\big)(x) \cdot \phi^{(N)}(x)^{-1} \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathrm{Ad}_{\phi^{(n)}(x)} \big(M_\phi \circ F_n\big)(x).\end{aligned}$$

von Neumann's ergodic theorem implies that $P_\phi \in \mathscr{B}\big(\mathrm{L}^2(X, \mathfrak{g})\big)$ is the orthogonal projection onto $\ker(1 - W_\phi)$ with W_ϕ the unitary operator

$$\big(W_\phi f\big)(x) := \mathrm{Ad}_{\phi(x)}(f \circ F_1)(x), \quad f \in \mathrm{L}^2(X, \mathfrak{g}),\ \mu_X\text{-almost every } x \in X.$$

The degree $P_\phi M_\phi$ transforms in a natural way under Lie group homomorphisms and under the relation of C^1-cohomology.

Degree under homomorphisms.

If $\phi = h \circ \delta$ with $h : G' \to G$ a Lie group homomorphism and $\delta \in C^1(X, G')$, then

$$P_\phi M_\phi = (\mathrm{d}h)_{e_{G'}}\big((P_\delta M_\delta)(\cdot)\big)$$

with $(\mathrm{d}h)_{e_{G'}} : \mathfrak{g}' \to \mathfrak{g}$ the differential of h and $\mathfrak{g}'$ the Lie algebra of G'.

Degree under C^1-cohomology.

If $\zeta, \delta \in C^1(X, G)$ are such that

$$\phi(x) = \zeta(x)^{-1}\,\delta(x)\,(\zeta \circ F_1)(x), \quad x \in X,$$

then $P_\delta M_\delta = \mathrm{Ad}_\zeta(P_\phi M_\phi)$.

$\pi(G)$ is a Lie group with Lie algebra $\mathfrak{g}_\pi$ and $\pi : G \to \pi(G) \subset \mathrm{U}(d_\pi)$ is a Lie group homomorphism. So, we obtain

$$P_{\pi\circ\phi} M_{\pi\circ\phi} = (\mathrm{d}\pi)_{e_G}\big((P_\phi M_\phi)(\cdot)\big),$$

and the function $P_{\pi\circ\phi} M_{\pi\circ\phi} : X \to \mathfrak{g}_\pi$ is the degree of $\pi \circ \phi$.

The degree of $\pi \circ \phi$ is the image of the degree of ϕ under the differential (pushforward) $(\mathrm{d}\pi)_{e_G} : \mathfrak{g} \to \mathfrak{g}_\pi$

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\ (\mathrm{d}\pi)_{e_G}\ } & \mathfrak{g}_\pi \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow{\ \pi\ } & \pi(G) \end{array}$$

$P_\phi M_\phi$ and $P_{\pi\circ\phi} M_{\pi\circ\phi}$ take simple forms in two particular cases.

Lemma (F_1 uniquely ergodic and $\pi\circ\phi$ diagonal)

Assume that $\phi \in C^1(X, G)$, that F_1 is uniquely ergodic, and that $\pi\circ\phi$ is diagonal for each $\pi \in \widehat{G}$. Then,

$$\lim_{N\to\infty} \left\| \frac{1}{N} \sum_{n=0}^{N-1} W_{\pi\circ\phi}^n M_{\pi\circ\phi} - (\mathrm{d}\pi)_{e_G}(M_{\phi,\star}) \right\|_{\mathsf{L}^\infty(X,\mathscr{B}(\mathbb{C}^{d_\pi}))} = 0,$$

with

$$M_{\phi,\star} := \int_X \mathrm{d}\mu_X(x)\, M_\phi(x).$$

Furthermore, $P_\phi M_\phi = M_{\phi,\star}$.

Lemma (T_ϕ uniquely ergodic)

Assume that $\phi \in C^1(X, G)$ *and that* T_ϕ *is uniquely ergodic. Then,*

$$\lim_{N\to\infty} \left\| \frac{1}{N} \sum_{n=0}^{N-1} W^n_{\pi\circ\phi} M_{\pi\circ\phi} - (\mathrm{d}\pi)_{e_G}(M_{\phi,\star}) \right\|_{\mathrm{L}^\infty(X,\mathscr{B}(\mathbb{C}^{d_\pi}))} = 0,$$

with

$$M_{\phi,\star} := \int_{X\times G} \mathrm{d}(\mu_X \otimes \mu_G)(x,g)\, \mathrm{Ad}_g M_\phi(x).$$

Furthermore, $P_\phi M_\phi = M_{\phi,\star}$.

Comment on the proofs.

The convergences in the L^∞-norm follow from the unique ergodicity of F_1 and T_ϕ. In the first lemma, $M_{\phi,\star}$ is simpler because $\pi \circ \phi$ is diagonal, so that $W^n_{\pi\circ\phi} M_{\pi\circ\phi} = M_{\pi\circ\phi} \circ F_n$. □

In the case T_ϕ uniquely ergodic and G connected, we have

$$\begin{aligned} M_{\phi,\star} &= \int_{X\times G} \mathrm{d}(\mu_X \otimes \mu_G)(x,g)\,\mathrm{Ad}_g M_\phi(x) \\ &= \int_G \mathrm{d}\mu_G(g)\,\mathrm{Ad}_g \left(\int_X \mathrm{d}\mu_X(x)\, M_\phi(x)\right) \\ &\in z(\mathfrak{g}), \end{aligned}$$

with

$$z(\mathfrak{g}) := \{Z \in \mathfrak{g} \mid [X,Z]_{\mathfrak{g}} = 0 \text{ for all } X \in \mathfrak{g}\}$$

the center of $\mathfrak{g}$.

Connected semisimple groups G have trivial center $z(\mathfrak{g}) = \{0\}$. Thus, there is no uniquely ergodic T_ϕ with nonzero degree if G is connected and semisimple (for example $G = \mathrm{SU}(n)$ or $G = \mathrm{SO}(n+1,\mathbb{R})$ with $n \geq 2$).

Mixing

By applying an abstract commutator criterion for mixing, we get:

Theorem (Mixing property of $U_{\phi,\pi,j}$)

Assume that $\phi \in C^1(X, G)$. *Then, the degree exists and is equal to* $D_{\phi,\pi} = i(\mathrm{d}\pi)_{e_G}\big((P_\phi M_\phi)(\,\cdot\,)\big)$, *and*

(a) $\lim_{N\to\infty} \big\langle \varphi, \big(U_{\phi,\pi,j}\big)^N \psi \big\rangle = 0$ *for each* $\varphi \in \ker(D_{\phi,\pi})^\perp$ *and* $\psi \in \mathcal{H}_j^{(\pi)}$,

(b) $U_{\phi,\pi,j}\big|_{\ker(D_{\phi,\pi})^\perp}$ *has purely continuous spectrum.*

Summing up the results for each π, one gets that U_ϕ is mixing in the subspace

$$\mathcal{H}_{\mathrm{mix}} := \bigoplus_{\pi \in \widehat{G}} \bigoplus_{j=1}^{d_\pi} \ker(D_{\phi,\pi})^\perp \subset \mathcal{H}.$$

If F_1 uniquely ergodic and $\pi \circ \phi$ diagonal, or if T_ϕ is uniquely ergodic, the theorem simplifies to:

Corollary

Assume that $\phi \in C^1(X, G)$ and suppose that F_1 is uniquely ergodic and $\pi \circ \phi$ diagonal, or that T_ϕ is uniquely ergodic. Then, the degree exists and is equal to $D_{\phi,\pi} = i(\mathrm{d}\pi)_{e_G}(M_{\phi,\star})$, and

(a) $\lim_{N\to\infty} \langle \varphi, (U_{\phi,\pi,j})^N \psi \rangle = 0$ *for each* $\varphi \in \ker(D_{\phi,\pi})^\perp$ *and* $\psi \in \mathcal{H}_j^{(\pi)}$,

(b) $U_{\phi,\pi,j}\big|_{\ker(D_{\phi,\pi})^\perp}$ *has purely continuous spectrum.*

In this case, $D_{\phi,\pi}$ is the multiplication operator by the constant matrix $i(\mathrm{d}\pi)_{e_G}(M_{\phi,\star})$. Thus, $\ker(D_{\phi,\pi})^\perp$ is easy to compute.

Absolutely continuous spectrum

Let

$$a_{\phi,\pi} := \operatorname*{ess\,inf}_{x\in X} \inf_{v\in\mathbb{C}^{d_\pi},\, \|v\|_{\mathbb{C}^{d_\pi}}=1} \left\langle v, \big(i(\mathrm{d}\pi)_{e_G}\big((P_\phi M_\phi)(x)\big)\big)^2 v \right\rangle_{\mathbb{C}^{d_\pi}} .$$

By applying an abstract commutator criterion for absolutely continuous spectrum, we get:

Theorem (Absolutely continuous spectrum of $U_{\phi,\pi,j}$)

Assume that

(i) $\phi \in C^1(X,G)$ + some more regularity,

(ii) $\lim_{N\to\infty} \left\| \frac{1}{N} \sum_{n=0}^{N-1} W^n_{\pi\circ\phi} M_{\pi\circ\phi} - (\mathrm{d}\pi)_{e_G}\big((P_\phi M_\phi)(\,\cdot\,)\big) \right\|_{\mathsf{L}^\infty(X,\mathscr{B}(\mathbb{C}^{d_\pi}))} = 0,$

(iii) $a_{\phi,\pi} > 0$.

Then, $U_{\phi,\pi,j}$ has purely absolutely continuous spectrum.

Summing up the results for each π, one gets that U_ϕ has purely absolutely continuous spectrum in a subspace

$$\mathcal{H}_{\mathsf{ac}} \subset \mathcal{H}_{\mathsf{mix}} \subset \mathcal{H}.$$

If F_1 uniquely ergodic and $\pi \circ \phi$ diagonal, or if T_ϕ is uniquely ergodic, the theorem simplifies to:

Corollary

Assume that

(i) $\phi \in C^1(X, G) +$ some more regularity,

(ii) F_1 *is uniquely ergodic and* $\pi \circ \phi$ *diagonal, or* T_ϕ *is uniquely ergodic,*

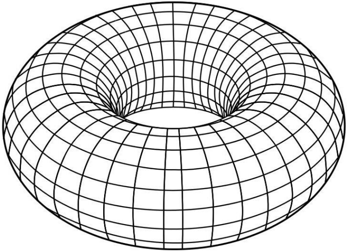
(iii) $\det\big((\mathrm{d}\pi)_{e_G}(M_{\phi,\star})\big) \neq 0.$

Then, $U_{\phi,\pi,j}$ *has purely absolutely continuous spectrum.*

In summary, in each subspace $\mathcal{H}_j^{(\pi)}$ we use commutator methods to determine the spectral properties of $U_{\phi,\pi,j} = U_\phi\big|_{\mathcal{H}_j^{(\pi)}}$.

$\mathcal{H}_1^{(\pi_4)}$	$\mathcal{H}_2^{(\pi_4)}$	$\mathcal{H}_3^{(\pi_4)}$	$\mathcal{H}_4^{(\pi_4)}$
$\mathcal{H}_1^{(\pi_3)}$	$\mathcal{H}_2^{(\pi_3)}$	$\mathcal{H}_3^{(\pi_3)}$	$\mathcal{H}_4^{(\pi_3)}$
$\mathcal{H}_1^{(\pi_2)}$	$\mathcal{H}_2^{(\pi_2)}$	$\mathcal{H}_3^{(\pi_2)}$	$\mathcal{H}_4^{(\pi_2)}$
$\mathcal{H}_1^{(\pi_1)}$	$\mathcal{H}_2^{(\pi_1)}$	$\mathcal{H}_3^{(\pi_1)}$	$\mathcal{H}_4^{(\pi_1)}$

Cocycles with values in a torus



Assume that $G = \mathbb{T}^d := (\mathbb{S}^1)^d$ $(d \in \mathbb{N}^*)$. Then, $\mathfrak{g} = i\,\mathbb{R}^d$, each $\pi^{(q)} \in \widehat{\mathbb{T}^d}$ is a character of $\mathbb{T}^d$ given by

$$\pi^{(q)}(z) := z_1^{q_1} \cdots z_d^{q_d}, \quad z = (z_1, \ldots, z_d) \in \mathbb{T}^d,\ q = (q_1, \ldots, q_d) \in \mathbb{Z}^d,$$

and

$$\mathcal{H}^{(q)} := \mathcal{H}_1^{(\pi^{(q)})} = \mathsf{L}^2(X, \mu_X) \otimes \mathbb{C}\pi^{(q)}.$$

If $\phi \in C^1(X, \mathbb{T}^d)$, then

$$D_{\phi,q} := D_{\phi,\pi^{(q)}} = i\left(\mathrm{d}\pi^{(q)}\right)_e\left((P_\phi M_\phi)(\cdot)\right)$$

with $e = e_{\mathbb{T}^d} = (1, \ldots, 1)$, and U_ϕ is mixing in the subspace

$$\mathcal{H}_{\mathrm{mix}} := \bigoplus_{q \in \mathbb{Z}^d} \ker(D_{\phi,q})^\perp \subset \mathcal{H}.$$

To say more on U_ϕ, we further assume that F_1 is uniquely ergodic and an additional regularity assumption of Dini-type on $\mathscr{L}_Y(\pi^{(q)} \circ \phi)$.

The unique ergodicity of F_1 implies that

$$D_{\phi,q} = i\big(\mathrm{d}\pi^{(q)}\big)_e(M_{\phi,\star}) \in \mathbb{R}$$

with

$$M_{\phi,\star} = \int_X \mathrm{d}\mu_X(x)\, M_\phi(x) \in i\,\mathbb{R}^d.$$

The last assumption of the corollary $\det\big((\mathrm{d}\pi)_{e_G}(M_{\phi,\star})\big) \neq 0$ is equivalent to $D_{\phi,q} \neq 0$. Thus, we obtain that U_ϕ has purely absolutely continuous spectrum in the subspace

$$\mathcal{H}_{\mathrm{ac}} := \bigoplus_{q\in\mathbb{Z}^d,\, D_{\phi,q}\neq 0} \mathcal{H}^{(q)} \subset \mathcal{H}_{\mathrm{mix}}.$$

Example

If $X = G = \mathbb{T}$, $\phi(x) = x^m$ $(m \in \mathbb{Z})$, *and*

$$F_t(x) := x\,\mathrm{e}^{2\pi i t\alpha}, \quad t \in \mathbb{R},\ x \in \mathbb{T},\ \alpha \in \mathbb{R} \setminus \mathbb{Q},$$

then $\phi \in C^\infty(\mathbb{T}, \mathbb{T})$, F_1 *is uniquely ergodic (irrational rotation), the degree of* ϕ *is*

$$\begin{aligned} M_{\phi,\star} &= \int_{\mathbb{T}} \mathrm{d}\mu_{\mathbb{T}}(x)\, M_\phi(x) \\ &= \int_{\mathbb{T}} \mathrm{d}\mu_{\mathbb{T}}(x) \left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \phi\big(x\,\mathrm{e}^{2\pi i t\alpha}\big) \right) \phi(x)^{-1} \\ &= \int_{\mathbb{T}} \mathrm{d}\mu_{\mathbb{T}}(x) \left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} x^m\,\mathrm{e}^{2m\pi i t\alpha} \right) x^{-m} \\ &= 2m\pi i\alpha, \end{aligned}$$

Example (continued)

and the degree of $\pi^{(q)} \circ \phi$ *is*

$$\left(\mathrm{d}\pi^{(q)}\right)_e(M_{\phi,\star}) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \pi^{(q)}\left(\mathrm{e}^{tM_{\phi,\star}}\right) = 2mq\pi i\alpha.$$

Therefore, if $m \neq 0$*, we obtain that* U_ϕ *has purely absolutely continuous spectrum in the subspace*

$$\begin{aligned}\mathcal{H}_{\mathrm{ac}} &= \bigoplus_{q\in\mathbb{Z}\setminus\{0\}} \mathcal{H}^{(q)} \\ &= \bigoplus_{q\in\mathbb{Z}\setminus\{0\}} \mathsf{L}^2(X,\mu_X) \otimes \mathbb{C}\pi^{(q)} \\ &= \left\{ \begin{array}{c} \text{orthocomplement of the functions} \\ \text{depending only on the first variable} \end{array} \right\}.\end{aligned}$$

Cocycles with values in U(2)

Assume that

$$G = \mathsf{U}(2) = \left\{ \begin{pmatrix} z_1 & z_2 \\ -\mathrm{e}^{i\theta}\,\overline{z_2} & \mathrm{e}^{i\theta}\,\overline{z_1} \end{pmatrix} \mid \theta \in [0, 2\pi),\ z_1, z_2 \in \mathbb{C},\ |z_1|^2 + |z_2|^2 = 1 \right\},$$

$$\mathfrak{g} = \mathfrak{u}(2) = \left\{ \begin{pmatrix} is_1 & z \\ -\overline{z} & is_2 \end{pmatrix} \mid s_1, s_2 \in \mathbb{R},\ z \in \mathbb{C} \right\}.$$

Using the representation theory for SU(2) and the epimorphism

$$\mathbb{T} \times \mathsf{SU}(2) \ni (z, g) \mapsto z\,g \in \mathsf{U}(2),$$

we can determine all the representations $\pi^{(\ell,m)}$ of U(2),

$$\pi^{(\ell,m)} : \mathsf{U}(2) \to \mathsf{U}(\ell+1), \quad \ell \in \mathbb{N},\ m \in \mathbb{Z}.$$

If $\phi \in C^1\big(X; \mathrm{U}(2)\big)$, then

$$D_{\phi,\ell,m} := D_{\phi,\pi^{(\ell,m)}} = i\big(\mathrm{d}\pi^{(\ell,m)}\big)_{I_2}\big((P_\phi M_\phi)(\cdot)\big)$$

with $I_2 := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = e_{\mathrm{U}(2)}$, and U_ϕ is mixing in the subspace

$$\bigoplus_{m\in\mathbb{Z}} \bigoplus_{\ell\in\mathbb{N}} \bigoplus_{j\in\{0,\dots,\ell\}} \ker(D_{\phi,\ell,m})^\perp \subset \mathcal{H}.$$

To say more on U_ϕ, we further assume that T_ϕ is uniquely ergodic. In this case, we have $D_{\phi,\ell,m} = i\big(\mathrm{d}\pi^{(\ell,m)}\big)_{I_2}(M_{\phi,\star})$ with

$$M_{\phi,\star} \in z\big(\mathfrak{u}(2)\big) = \big\{isI_2 \mid s \in \mathbb{R}\big\}.$$

Therefore, $M_{\phi,\star} = is_\phi I_2$ for some $s_\phi \in \mathbb{R}$.

Using an explicit formula for $\pi^{(\ell,m)}$, we obtain

$$i\big(\mathrm{d}\pi^{(\ell,m)}\big)_{I_2}(M_{\phi,\star})_{j,k} = -s_\phi\,(2m-\ell)\,j\,!(\ell-j)!\,\delta_{j,k}, \quad j,k \in \{0,\dots,\ell\}.$$

(constant diagonal matrix)

Thus, if $s_\phi \neq 0$, U_ϕ is mixing in the subspace

$$\mathcal{H}_{\mathrm{mix}} := \bigoplus_{m\in\mathbb{Z}} \bigoplus_{\ell\in\mathbb{N}\setminus\{2m\}} \bigoplus_{j\in\{0,\dots,\ell\}} \mathcal{H}_j^{(\pi^{(\ell,m)})} \subset \mathcal{H}.$$

Under an additional regularity assumption of Dini-type on $\mathscr{L}_Y(\pi^{(\ell,m)}\circ\phi)$, we obtain that U_ϕ has purely absolutely continuous spectrum in $\mathcal{H}_{\mathrm{mix}}$.

Example

Using the isomorphism

$$\mathrm{SO}(3,\mathbb{R}) \times \mathbb{T} \simeq \mathrm{U}(2)$$

and results of Eliasson and Hou on skew products on $\mathbb{T}^d \times \mathrm{SO}(3,\mathbb{R})$*, we can produce skew-products* T_ϕ *on* $\mathbb{T}^d \times \mathrm{U}(2)$ *satisfying our assumptions.*

Namely, skew-products T_ϕ *with* $\phi \in C^\infty\left(\mathbb{T}^d; \mathrm{U}(2)\right)$, T_ϕ *uniquely ergodic, and nonzero degree* $M_{\phi,\star} = is_\phi I_2$.

Thank you !

References

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