Degree, mixing, and absolutely continuous spectrum of cocycles with values in compact Lie groups

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Santiago, June 2016

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Commutator methods for unitary operators

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- A, self-adjoint operator in $\mathcal{H}$ with domain $\mathcal{D}(A)$

Definition

$S \in \mathscr{B}(\mathcal{H})$ satisfies $S \in C^{+0}(A)$ if

$$\int_0^1 \frac{\mathrm{d}t}{t}\,\big\|\,\mathrm{e}^{-itA}\,S\,\mathrm{e}^{itA} - S\big\| < \infty.$$

(Dini-type regularity along the "flow" of A)

Definition

$S \in \mathscr{B}(\mathcal{H})$ satisfies $S \in C^k(A)$ if

$$\mathbb{R} \ni t \mapsto \mathrm{e}^{-itA}\, S\, \mathrm{e}^{itA} \in \mathscr{B}(\mathcal{H})$$

is strongly of class C^k.

$S \in C^1(A)$ if and only if

$$\big|\langle A\varphi, S\varphi\rangle - \langle \varphi, SA\varphi\rangle\big| \leq \text{Const.}\, \|\varphi\|^2 \quad \text{for all } \varphi \in \mathcal{D}(A).$$

The bounded operator associated to the continuous extension of the quadratic form is denoted by $[S, A]$, and

$$[iS, A] = \text{s-}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{e}^{-itA}\, S\, \mathrm{e}^{itA} \in \mathscr{B}(\mathcal{H}).$$

Definition

$S \in \mathscr{B}(\mathcal{H})$ satisfies $S \in C^{1+0}(A)$ if $S \in C^1(A)$ and $[A, S] \in C^{+0}(A)$.

We have the inclusions

$$C^2(A) \subset C^{1+0}(A) \subset C^1(A) \subset C^{+0}(A) \subset C^0(A) = \mathscr{B}(\mathcal{H}).$$

Theorem (Fernández-Richard-T. 2013)

Let U be a unitary operator in $\mathcal{H}$ and let A be a self-adjoint operator in $\mathcal{H}$ with $U \in C^{1+0}(A)$. Suppose there is an open set $\Theta \subset \mathbb{S}^1$, a number $a > 0$, and $K \in \mathscr{K}(\mathcal{H})$ such that

$$E^U(\Theta)U^{-1}[A, U]E^U(\Theta) \geq aE^U(\Theta) + K. \qquad (\bigstar)$$

Then, U has at most finitely many eigenvalues in Θ (multiplicities counted), and U has no singular continuous spectrum in Θ.

- The inequality $(\bigstar)$ is called a Mourre estimate for U on Θ.
- The operator A is called a conjugate operator for U on Θ.
- If $K = 0$, U has purely absolutely continuous spectrum in $\Theta \cap \sigma(U)$.

Criterion for the mixing property of U :

Theorem (T. 2015, Richard-T. 2016)

Let U be a unitary operator in $\mathcal{H}$ and let A be a self-adjoint operator in $\mathcal{H}$ with $U \in C^1(A)$. Assume that

$$D := \operatorname*{s-lim}_{N\to\infty} \frac{1}{N}\big[A, U^N\big]\, U^{-N}$$

exists. Then,

(a) $\lim_{N\to\infty} \langle \varphi, U^N \psi \rangle = 0$ *for each* $\varphi \in \ker(D)^\perp$ *and* $\psi \in \mathcal{H}$,

(b) $U|_{\ker(D)^\perp}$ *has purely continuous spectrum.*

- D is bounded and self-adjoint because it is the strong limit of bounded self-adjoint operators.
- $DU^n = U^n D$ for each $n \in \mathbb{Z}$.
- Point (b) is a simple consequence of point (a).

Let's determine conditions under which U has purely absolutely continuous spectrum in $\ker(D)^{\perp}$.

If $D \in C^1(A)$, the operators

$$A_D := AD + DA \quad \text{and} \quad A_{D,N} := \frac{1}{N}\sum_{n=0}^{N-1} U^n A_D U^{-n}, \quad N \in \mathbb{N}^*,$$

are essentially self-adjoint on $\mathcal{D}(A)$, and we have:

Proposition (Mourre estimate)

Let U be a unitary operator in $\mathcal{H}$ and let A be a self-adjoint operator in $\mathcal{H}$ with $U \in C^1(A)$. Assume that $D = \text{u-lim}_{N\to\infty} \frac{1}{N}[A, U^N]U^{-N}$ exists and satisfies $D \in C^1(A)$. Then, for each $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}^*$ such that

$$U^{-1}\big[A_{D,N}, U\big] \geq 2D^2 - \varepsilon \quad \textit{for } N \geq N_\varepsilon.$$

Idea of the proof.

Using the relation $[D, U] = 0$, we get

$$\begin{aligned}\big[A_{D,N}, U\big] &= \frac{1}{N}\sum_{n=0}^{N-1} U^n[AD + DA, U]U^{-n} \\ &= \frac{1}{N}\sum_{n=0}^{N-1} U^n\big([A, U]D + D[A, U]\big)U^{-n}.\end{aligned}$$

Thus, if we set

$$D_N := \frac{1}{N}\big[A, U^N\big]U^{-N} = \frac{1}{N}\sum_{n=0}^{N-1} U^n\big([A, U]U^{-1}\big)U^{-n},$$

Idea of the proof (continued).

we obtain

$$\begin{aligned}
U^{-1}\big[A_{D,N}, U\big] &= U^{-1}\left(\frac{1}{N}\sum_{n=0}^{N-1} U^n\big([A,U]U^{-1}D + D[A,U]U^{-1}\big)U^{-n}\right)U \\
&= U^{-1}\big(D_N D + D D_N\big)U \\
&= 2D^2 + U^{-1}\big(\underbrace{(D_N - D)}_{\to 0 \text{ in norm}} D + D \underbrace{(D_N - D)}_{\to 0 \text{ in norm}}\big)U.
\end{aligned}$$

□

With what precedes, we obtain a criterion for the absolute continuity of U in $\ker(D)^{\perp}$:

Theorem (Absolutely continuous spectrum)

Let U be a unitary operator in $\mathcal{H}$ and let A be a self-adjoint operator in $\mathcal{H}$ with $U \in C^1(A)$. Assume that $D = \text{u-lim}_{N\to\infty} \frac{1}{N}\left[A, U^N\right] U^{-N}$ exists, that $D \in C^1(A)$, that $[A, U] \in C^{+0}(A_D)$, and that

$$D^2 E^U(\Theta) \geq c_\Theta E^U(\Theta) \quad \text{for some open set } \Theta \subset \mathbb{S}^1 \text{ and } c_\Theta > 0.$$

Then, U has purely absolutely continuous spectrum in $\Theta \cap \sigma(U)$.

- The conditions $U \in C^1(A)$ and $[A, U] \in C^{+0}(A_D)$ imply the regularity condition $U \in C^{1+0}(A_{D,N})$.
- The theorem gives a result in $\ker(D)^{\perp}$ because $D^2 = D = 0$ on $\ker(D)$.

Summing up, we have

$$\boxed{U \in C^1(A) \text{ and } D = \text{s-lim}_{N\to\infty} \frac{1}{N}\left[A, U^N\right] U^{-N} \text{ exists}} \quad (\bigstar)$$

$$\Downarrow$$

$$\boxed{U \text{ mixing in } \ker(D)^\perp}$$

and

$$\boxed{(\bigstar) + \text{additional regularity conditions} + \text{positivity assumption}}$$

$$\Downarrow$$

$$\boxed{U \text{ purely absolutely continuous in } \ker(D)^\perp}$$

Cocycles with values in compact Lie groups

- X, compact manifold with probability measure μ_X
- $\{F_t\}_{t\in\mathbb{R}}$, C^1 measure-preserving flow on X with Lie derivative $\mathscr{L}_Y$
- G, compact Lie group with Haar measure μ_G, identity e_G, Lie algebra $\mathfrak{g}$, and Lie bracket $[\cdot,\cdot]_{\mathfrak{g}} : \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$

A measurable function $\phi : X \to G$ induces a measurable cocycle over F_1

$$X \times \mathbb{Z} \ni (x, n) \mapsto \phi^{(n)}(x) \in G$$

given by

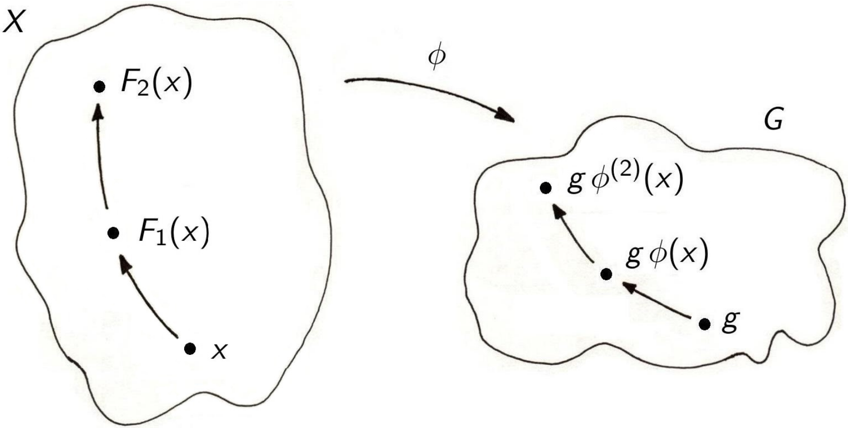
$$\phi^{(n)}(x) := \begin{cases} \phi(x)(\phi\circ F_1)(x)\cdots(\phi\circ F_{n-1})(x) & \text{if } n \geq 1 \\ e_G & \text{if } n = 0 \\ \left(\phi^{(-n)}\circ F_n\right)(x)^{-1} & \text{if } n \leq -1. \end{cases}$$

The skew product associated to ϕ is the measure preserving map

$$T_\phi : X \times G \to X \times G, \quad (x, g) \mapsto \big(F_1(x), g\,\phi(x)\big),$$

with iterates

$$T_\phi^n(x, g) = \big(F_n(x), g\,\phi^{(n)}(x)\big), \quad n \in \mathbb{Z}.$$



The Koopman operator for T_ϕ is the unitary operator

$$U_\phi \psi := \psi \circ T_\phi, \quad \psi \in \mathcal{H} := \mathsf{L}^2(X \times G, \mu_X \otimes \mu_G).$$

Peter-Weyl's theorem gives an orthogonal decomposition

$$\mathcal{H} = \bigoplus_{\pi \in \widehat{G}} \bigoplus_{j=1}^{d_\pi} \mathcal{H}_j^{(\pi)}, \quad \mathcal{H}_j^{(\pi)} := \bigoplus_{k=1}^{d_\pi} \mathsf{L}^2(X, \mu_X) \otimes \{\pi_{jk}\},$$

with $\pi : G \to \mathsf{U}(d_\pi)$ the finite-dimensional irreducible unitary representations of G, and $U_{\phi,\pi,j} := U_\phi\big|_{\mathcal{H}_j^{(\pi)}}$ the restriction given by

$$U_{\phi,\pi,j} \sum_{k=1}^{d_\pi} \varphi_k \otimes \pi_{jk} = \sum_{k,\ell=1}^{d_\pi} \big(\varphi_k \circ F_1\big)\big(\pi_{\ell k} \circ \phi\big) \otimes \pi_{j\ell}, \quad \varphi_k \in \mathsf{L}^2(X, \mu_X).$$

(later, we will apply commutator methods to $U_{\phi,\pi,j}$ in $\mathcal{H}_j^{(\pi)}$)

Degree of a cocycle

Definition (Degree of ϕ)

Assume that $\phi \in C^1(X, G)$ and let $M_\phi := \mathscr{L}_Y \phi \cdot \phi^{-1} \in C(X, \mathfrak{g})$. Then, we define the degree of ϕ as the function $P_\phi M_\phi : X \to \mathfrak{g}$ given by

$$(P_\phi M_\phi)(x) := \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathrm{Ad}_{\phi^{(n)}(x)}\big(M_\phi \circ F_n\big)(x), \ \ \mu_X\text{-almost every } x \in X.$$

We have $\big(P_\phi M_\phi\big)(x) := \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \big(W_\phi^n M_\phi\big)(x)$ with W_ϕ the unitary operator in $\mathrm{L}^2(X, \mathfrak{g})$ given by

$$\big(W_\phi f\big)(x) := \mathrm{Ad}_{\phi(x)}(f \circ F_1)(x), \quad f \in \mathrm{L}^2(X, \mathfrak{g}).$$

Thus, von Neumann's ergodic theorem implies that $P_\phi \in \mathscr{B}\big(\mathrm{L}^2(X, \mathfrak{g})\big)$ is the orthogonal projection onto $\ker(1 - W_\phi)$.

The degree $P_\phi M_\phi$ transforms in a natural way under Lie group homomorphisms and under the relation of C^1-cohomology.

Degree under homomorphisms.

If $\phi = h \circ \delta$ with $h : G' \to G$ a Lie group homomorphism and $\delta \in C^1(X, G')$, then

$$P_\phi M_\phi = (\mathrm{d}h)_{e_{G'}}\big((P_\delta M_\delta)(\,\cdot\,)\big)$$

with $(\mathrm{d}h)_{e_{G'}} : \mathfrak{g}' \to \mathfrak{g}$ the differential of h and $\mathfrak{g}'$ the Lie algebra of G'.

Degree under C^1-cohomology.

If $\zeta, \delta \in C^1(X, G)$ are such that

$$\phi(x) = \zeta(x)^{-1}\,\delta(x)\,(\zeta \circ F_1)(x), \quad x \in X,$$

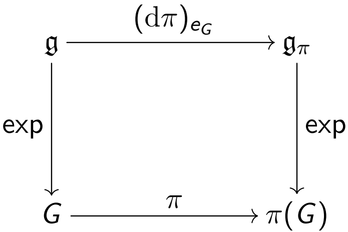
then $P_\delta M_\delta = \mathrm{Ad}_\zeta(P_\phi M_\phi)$.

$\pi(G)$ is a Lie group with Lie algebra $\mathfrak{g}_\pi$ and $\pi : G \to \pi(G) \subset \mathrm{U}(d_\pi)$ is a Lie group homomorphism. So, we obtain

$$P_{\pi\circ\phi} M_{\pi\circ\phi} = (\mathrm{d}\pi)_{e_G}\big((P_\phi M_\phi)(\cdot)\big),$$

and the function $P_{\pi\circ\phi} M_{\pi\circ\phi} : X \to \mathfrak{g}_\pi$ is the degree of $\pi \circ \phi$.

The degree of $\pi \circ \phi$ is the image of the degree of ϕ under the differential (pushforward) $(\mathrm{d}\pi)_{e_G} : \mathfrak{g} \to \mathfrak{g}_\pi$



$P_\phi M_\phi$ and $P_{\pi\circ\phi} M_{\pi\circ\phi}$ take a simple form in two particular cases.

Lemma (F_1 uniquely ergodic and $\pi \circ \phi$ diagonal)

Assume that $\phi \in C^1(X, G)$, that F_1 is uniquely ergodic, and that $\pi \circ \phi$ is diagonal for each $\pi \in \widehat{G}$. Then,

$$\lim_{N\to\infty} \left\| \frac{1}{N} \sum_{n=0}^{N-1} W_{\pi\circ\phi}^n M_{\pi\circ\phi} - (\mathrm{d}\pi)_{e_G}(M_{\phi,\star}) \right\|_{\mathsf{L}^\infty(X,\mathscr{B}(\mathbb{C}^{d_\pi}))} = 0,$$

with

$$M_{\phi,\star} := \int_X \mathrm{d}\mu_X(x)\, M_\phi(x).$$

Furthermore, $P_\phi M_\phi = M_{\phi,\star}$.

Lemma (T_ϕ uniquely ergodic)

Assume that $\phi \in C^1(X, G)$ and that T_ϕ is uniquely ergodic. Then,

$$\lim_{N\to\infty} \left\| \frac{1}{N} \sum_{n=0}^{N-1} W_{\pi\circ\phi}^n M_{\pi\circ\phi} - (\mathrm{d}\pi)_{e_G}(M_{\phi,\star}) \right\|_{\mathrm{L}^\infty(X,\mathscr{B}(\mathbb{C}^{d_\pi}))} = 0,$$

with

$$M_{\phi,\star} := \int_{X\times G} \mathrm{d}(\mu_X \otimes \mu_G)(x,g)\, \mathrm{Ad}_g M_\phi(x).$$

Furthermore, $P_\phi M_\phi = M_{\phi,\star}$.

Idea of the proofs.

The convergences in the L^∞-norm follow from the unique ergodicity of F_1 and T_ϕ. In the first lemma, $M_{\phi,\star}$ is simpler because $\pi \circ \phi$ is diagonal, so that $W_{\pi\circ\phi}^n M_{\pi\circ\phi} = M_{\pi\circ\phi} \circ F_n$. □

In the case T_ϕ uniquely ergodic and G connected, we have

$$\begin{aligned} M_{\phi,\star} &= \int_{X\times G} \mathrm{d}(\mu_X \otimes \mu_G)(x,g)\,\mathrm{Ad}_g M_\phi(x) \\ &= \int_G \mathrm{d}\mu_G(g)\,\mathrm{Ad}_g\left(\int_X \mathrm{d}\mu_X(x)\,M_\phi(x)\right) \\ &\in z(\mathfrak{g}), \end{aligned}$$

with

$$z(\mathfrak{g}) := \left\{Z \in \mathfrak{g} \mid [X,Z]_{\mathfrak{g}} = 0 \text{ for all } X \in \mathfrak{g}\right\}$$

the center of $\mathfrak{g}$.

Connected semisimple groups G have trivial center $z(\mathfrak{g}) = \{0\}$. Thus, there is no uniquely ergodic T_ϕ with nonzero degree if G is connected and semisimple (for example $G = \mathrm{SU}(n)$ or $G = \mathrm{SO}(n+1,\mathbb{R})$ with $n \geq 2$).

Mixing

To apply the abstract theorem on mixing to $U_{\phi,\pi,j}$ in $\mathcal{H}_j^{(\pi)}$, we first need an operator A :

Lemma (Definition of A)

The operator

$$A\sum_{k=1}^{d_\pi}\varphi_k\otimes\pi_{jk}:=\sum_{k=1}^{d_\pi}i\mathscr{L}_Y\varphi_k\otimes\pi_{jk},\quad \varphi_k\in C^1(X),$$

is essentially self-adjoint in $\mathcal{H}_j^{(\pi)}$, with closure denoted by the same symbol. Furthermore, if $\mathscr{L}_Y(\pi\circ\phi)\in \mathrm{L}^\infty\big(X,\mathscr{B}(\mathbb{C}^{d_\pi})\big)$, then $U_{\phi,\pi,j}\in C^1(A)$ with

$$\big[A,U_{\phi,\pi,j}\big]=iM_{\pi\circ\phi}\,U_{\phi,\pi,j}.$$

Next, we prove the existence of the corresponding strong limit $D_{\phi,\pi}$:

Lemma (Existence of $D_{\phi,\pi}$)

Assume that $\phi \in C^1(X, G)$. Then, the strong limit

$$D_{\phi,\pi} = \operatorname*{s-lim}_{N\to\infty} \frac{1}{N}\left[A, \left(U_{\phi,\pi,j}\right)^N\right]\left(U_{\phi,\pi,j}\right)^{-N}$$

exists and satisfies $D_{\phi,\pi} = i\,(\mathrm{d}\pi)_{e_G}\big((P_\phi M_\phi)(\,\cdot\,)\big)$.

The strong limit $D_{\phi,\pi}$ of the abstract theory is equal (as a multiplication operator in $\mathcal{H}_j^{(\pi)}$) to the degree of $\pi \circ \phi$:

$$P_{\pi\circ\phi} M_{\pi\circ\phi} = (\mathrm{d}\pi)_{e_G}\big((P_\phi M_\phi)(\,\cdot\,)\big).$$

Applying the abstract theorem on mixing, we get:

Theorem (Mixing property of $U_{\phi,\pi,j}$)

Assume that $\phi \in C^1(X, G)$. Then, the strong limit $D_{\phi,\pi}$ exists and is equal to $i\,(\mathrm{d}\pi)_{e_G}\big((P_\phi M_\phi)(\cdot)\big)$, and

(a) $\lim_{N\to\infty} \langle \varphi, \big(U_{\phi,\pi,j}\big)^N \psi\rangle = 0$ *for each* $\varphi \in \ker(D_{\phi,\pi})^\perp$ *and* $\psi \in \mathcal{H}_j^{(\pi)}$,

(b) $U_{\phi,\pi,j}\big|_{\ker(D_{\phi,\pi})^\perp}$ *has purely continuous spectrum.*

Summing up the results for each π, one gets that U_ϕ is mixing in the subspace

$$\mathcal{H}_{\mathrm{mix}} := \bigoplus_{\pi\in\widehat{G}} \bigoplus_{j=1}^{d_\pi} \ker(D_{\phi,\pi})^\perp \subset \mathcal{H}.$$

If F_1 uniquely ergodic and $\pi \circ \phi$ diagonal, or if T_ϕ is uniquely ergodic, the theorem simplifies to:

Corollary

Assume that $\phi \in C^1(X, G)$ and suppose that F_1 is uniquely ergodic and $\pi \circ \phi$ diagonal, or that T_ϕ is uniquely ergodic. Then, the strong limit $D_{\phi,\pi}$ exists and is equal to $i\,(\mathrm{d}\pi)_{e_G}(M_{\phi,\star})$, and

(a) $\lim_{N\to\infty} \langle \varphi, (U_{\phi,\pi,j})^N \psi \rangle = 0$ *for each* $\varphi \in \ker(D_{\phi,\pi})^\perp$ *and* $\psi \in \mathcal{H}_j^{(\pi)}$,

(b) $U_{\phi,\pi,j}\big|_{\ker(D_{\phi,\pi})^\perp}$ *has purely continuous spectrum.*

In this case, $D_{\phi,\pi}$ is the multiplication operator by the constant matrix $i\,(\mathrm{d}\pi)_{e_G}(M_{\phi,\star})$. Thus, $\ker(D_{\phi,\pi})^\perp$ is easy to determine.

Absolutely continuous spectrum

Let

$$a_{\phi,\pi} := \operatorname*{ess\,inf}_{x\in X} \inf_{v\in\mathbb{C}^{d_\pi},\, \|v\|_{\mathbb{C}^{d_\pi}}=1} \left\langle v, \big(i\,(\mathrm{d}\pi)_{e_G}\big((P_\phi M_\phi)(x)\big)\big)^2 v \right\rangle_{\mathbb{C}^{d_\pi}} .$$

Then, we have:

Theorem (Absolutely continuous spectrum of $U_{\phi,\pi,j}$)

Assume that

(i) $\phi \in C^1(X,G)$,

(ii) $\lim_{N\to\infty} \left\| \frac{1}{N}\sum_{n=0}^{N-1} W^n_{\pi\circ\phi} M_{\pi\circ\phi} - (\mathrm{d}\pi)_{e_G}\big((P_\phi M_\phi)(\,\cdot\,)\big) \right\|_{\mathsf{L}^\infty(X,\mathscr{B}(\mathbb{C}^{d_\pi}))} = 0,$

(iii) $\mathscr{L}_Y\big((\mathrm{d}\pi)_{e_G}\big((P_\phi M_\phi)(\,\cdot\,)\big)\big) \in \mathsf{L}^\infty\big(X, \mathscr{B}(\mathbb{C}^{d_\pi})\big),$

(iv) $M_{\pi\circ\phi} \in C^{+0}\big(A_{D_{\phi,\pi}}\big),$

(v) $a_{\phi,\pi} > 0.$

Then, $U_{\phi,\pi,j}$ has purely absolutely continuous spectrum.

Idea of the proof.

The proof consists in checking the assumptions of the abstract theorem of the beginning.

- The condition $U_{\phi,\pi,j} \in C^1(A)$ follows from the assumption $\phi \in C^1(X, G)$.
- The existence of the uniform limit

$$D_{\phi,\pi} = \underset{N\to\infty}{\text{u-lim}} \frac{1}{N}\left[A, \left(U_{\phi,\pi,j}\right)^N\right]\left(U_{\phi,\pi,j}\right)^{-N}$$

  follows from the assumption

$$\lim_{N\to\infty} \left\| \frac{1}{N}\sum_{n=0}^{N-1} W_{\pi\circ\phi}^n M_{\pi\circ\phi} - (\mathrm{d}\pi)_{e_G}\left((P_\phi M_\phi)(\,\cdot\,)\right) \right\|_{\mathrm{L}^\infty(X,\mathscr{B}(\mathbb{C}^{d_\pi}))} = 0.$$

Idea of the proof (continued).

- The condition $D_{\phi,\pi} \in C^1(A)$ follows from the assumption

$$\mathscr{L}_Y\big((\mathrm{d}\pi)_{e_G}\big((P_\phi M_\phi)(\,\cdot\,)\big)\big) \in \mathsf{L}^\infty\big(X, \mathscr{B}(\mathbb{C}^{d_\pi})\big).$$

- The condition $[A, U_{\phi,\pi,j}] \in C^{+0}(A_{D_{\phi,\pi}})$ follows from the assumption $M_{\pi\circ\phi} \in C^{+0}\big(A_{D_{\phi,\pi}}\big)$.
- The condition

$$D_{\phi,\pi}^2 E^{U_{\phi,\pi,j}}(\Theta) \geq c_\Theta E^{U_{\phi,\pi,j}}(\Theta) \quad (\text{with } \Theta = \mathbb{S}^1)$$

follows from the assumption $a_{\phi,\pi} > 0$.

□

Summing up the results for each π, one gets that U_ϕ has purely absolutely continuous spectrum in a subspace

$$\mathcal{H}_{\mathrm{ac}} \subset \mathcal{H}_{\mathrm{mix}} \subset \mathcal{H}.$$

If F_1 uniquely ergodic and $\pi \circ \phi$ diagonal, or if T_ϕ is uniquely ergodic, the theorem simplifies to:

Corollary

Assume that

(i) $\phi \in C^1(X, G)$,

(ii) F_1 *is uniquely ergodic and* $\pi \circ \phi$ *diagonal, or* T_ϕ *is uniquely ergodic,*

(iii) $M_{\pi\circ\phi} \in C^{+0}\left(A_{D_{\phi,\pi}}\right)$,

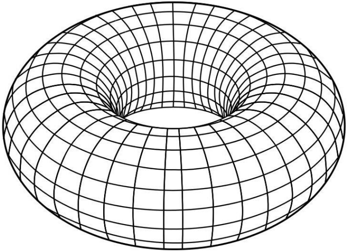
(iv) $\det\left((\mathrm{d}\pi)_{e_G}(M_{\phi,\star})\right) \neq 0$.

Then, $U_{\phi,\pi,j}$ *has purely absolutely continuous spectrum.*

In summary, in each subspace $\mathcal{H}_j^{(\pi)}$ we use a different limit $D_{\phi,\pi}$ and conjugate operator $A_{D_{\phi,\pi},N}$ to determine the spectral properties of $U_{\phi,\pi,j} = U_\phi\big|_{\mathcal{H}_j^{(\pi)}}$.

$\mathcal{H}_1^{(\pi_4)}$	$\mathcal{H}_2^{(\pi_4)}$	$\mathcal{H}_3^{(\pi_4)}$	$\mathcal{H}_4^{(\pi_4)}$
$\mathcal{H}_1^{(\pi_3)}$	$\mathcal{H}_2^{(\pi_3)}$	$\mathcal{H}_3^{(\pi_3)}$	$\mathcal{H}_4^{(\pi_3)}$
$\mathcal{H}_1^{(\pi_2)}$	$\mathcal{H}_2^{(\pi_2)}$	$\mathcal{H}_3^{(\pi_2)}$	$\mathcal{H}_4^{(\pi_2)}$
$\mathcal{H}_1^{(\pi_1)}$	$\mathcal{H}_2^{(\pi_1)}$	$\mathcal{H}_3^{(\pi_1)}$	$\mathcal{H}_4^{(\pi_1)}$

Cocycles with values in a torus



Assume that $G = \mathbb{T}^d := (\mathbb{S}^1)^d$ ($d \in \mathbb{N}^*$). Then, $\mathfrak{g} = i\,\mathbb{R}^d$, each $\pi^{(q)} \in \widehat{\mathbb{T}^d}$ is a character of $\mathbb{T}^d$ given by

$$\pi^{(q)}(z) := z_1^{q_1} \cdots z_d^{q_d}, \quad z = (z_1, \dots, z_d) \in \mathbb{T}^d,\ q = (q_1, \dots, q_d) \in \mathbb{Z}^d,$$

and

$$\mathcal{H}^{(q)} := \mathcal{H}_1^{(\pi^{(q)})} = \mathsf{L}^2(X, \mu_X) \otimes \left\{\pi^{(q)}\right\}.$$

If $\phi \in C^1(X, \mathbb{T}^d)$, then

$$D_{\phi,q} := D_{\phi,\pi^{(q)}} = \underset{N\to\infty}{\text{s-lim}} \frac{1}{N} \left[A, \left(U_{\phi,\pi^{(q)},1}\right)^N\right] \left(U_{\phi,\pi^{(q)},1}\right)^{-N}$$

exists and is equal to $i\left(\mathrm{d}\pi^{(q)}\right)_e\left((P_\phi M_\phi)(\cdot)\right)$ with $e = e_{\mathbb{T}^d} = (1, \ldots, 1)$, and U_ϕ is mixing in the subspace

$$\mathcal{H}_{\mathrm{mix}} := \bigoplus_{q\in\mathbb{Z}^d} \ker(D_{\phi,q})^\perp \subset \mathcal{H}.$$

To say more on U_ϕ, we further assume that F_1 is uniquely ergodic and

$$\int_0^1 \frac{\mathrm{d}t}{t} \left\|\mathscr{L}_Y(\pi^{(q)} \circ \phi) \circ F_t - \mathscr{L}_Y(\pi^{(q)} \circ \phi)\right\|_{\mathrm{L}^\infty(X,\mu_X)} < \infty, \quad q \in \mathbb{Z}^d.$$

The unique ergodicity of F_1 implies that

$$D_{\phi,q} = i\big(\mathrm{d}\pi^{(q)}\big)_e(M_{\phi,\star}) = i\int_X \mathrm{d}\mu_X(x)\, M_{\pi^{(q)}\circ\phi}(x) \in \mathbb{R}$$

with

$$M_{\phi,\star} = \int_X \mathrm{d}\mu_X(x)\, M_\phi(x) \in i\,\mathbb{R}^d \quad \text{and} \quad M_{\pi^{(q)}\circ\phi} = \mathscr{L}_Y(\pi^{(q)}\circ\phi)\cdot(\pi^{(q)}\circ\phi)^{-1}.$$

We also have

$$A_{D_{\phi,q}} = AD_{\phi,q} + D_{\phi,q}A = 2AD_{\phi,q},$$

which implies that $C^{+0}(A) \subset C^{+0}\big(A_{D_{\phi,q}}\big)$. Thus, the assumption $M_{\pi\circ\phi} \in C^{+0}\big(A_{D_{\phi,\pi}}\big)$ of the corollary reduces to $M_{\pi^{(q)}\circ\phi} \in C^{+0}(A)$, which follows from the Dini-type assumption above.

The last assumption of the corollary $\det\big((\mathrm{d}\pi)_{e_G}(M_{\phi,\star})\big) \neq 0$ is equivalent to $D_{\phi,q} \neq 0$. Thus, we obtain that U_ϕ has purely absolutely continuous spectrum in the subspace

$$\mathcal{H}_{\mathrm{ac}} := \bigoplus_{q\in\mathbb{Z}^d,\, D_{\phi,q}\neq 0} \mathcal{H}^{(q)} \subset \mathcal{H}_{\mathrm{mix}}.$$

Example

If $X = G = \mathbb{T}$, $\phi(x) = x^m$ $(m \in \mathbb{Z})$, *and*

$$F_t(x) := x\,\mathrm{e}^{2\pi i t\alpha}, \quad t \in \mathbb{R},\ x \in \mathbb{T},\ \alpha \in \mathbb{R} \setminus \mathbb{Q},$$

then $\phi \in C^\infty(\mathbb{T}, \mathbb{T})$, F_1 *is uniquely ergodic (irrational rotation), the degree of* ϕ *is*

$$\begin{aligned} M_{\phi,\star} &= \int_{\mathbb{T}} \mathrm{d}\mu_{\mathbb{T}}(x)\, M_\phi(x) \\ &= \int_{\mathbb{T}} \mathrm{d}\mu_{\mathbb{T}}(x) \left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \phi\big(x\,\mathrm{e}^{2\pi i t\alpha}\big) \right) \phi(x)^{-1} \\ &= \int_{\mathbb{T}} \mathrm{d}\mu_{\mathbb{T}}(x) \left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} x^m\,\mathrm{e}^{2m\pi i t\alpha} \right) x^{-m} \\ &= 2m\pi i\alpha, \end{aligned}$$

Example (continued)

and the degree of $\pi^{(q)} \circ \phi$ *is*

$$\left(\mathrm{d}\pi^{(q)}\right)_1(M_{\phi,\star}) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \pi^{(q)}\left(\mathrm{e}^{tM_{\phi,\star}}\right) = 2mq\pi i\alpha.$$

Therefore, if $m \neq 0$*, we obtain that* U_ϕ *has purely absolutely continuous spectrum in the subspace*

$$\begin{aligned}\mathcal{H}_{\mathrm{ac}} &= \bigoplus_{q\in\mathbb{Z}\setminus\{0\}} \mathcal{H}^{(q)} \\ &= \bigoplus_{q\in\mathbb{Z}\setminus\{0\}} \mathsf{L}^2(X,\mu_X)\otimes\left\{\pi^{(q)}\right\} \\ &= \left\{\begin{array}{c}\text{orthocomplement of the functions}\\ \text{depending only on the first variable}\end{array}\right\}.\end{aligned}$$

Cocycles with values in U(2)

Assume that

$$G = \mathsf{U}(2) = \left\{ \begin{pmatrix} z_1 & z_2 \\ -\mathrm{e}^{i\theta}\,\overline{z_2} & \mathrm{e}^{i\theta}\,\overline{z_1} \end{pmatrix} \mid \theta \in [0, 2\pi),\ z_1, z_2 \in \mathbb{C},\ |z_1|^2 + |z_2|^2 = 1 \right\},$$

$$\mathfrak{g} = \mathfrak{u}(2) = \left\{ \begin{pmatrix} is_1 & z \\ -\overline{z} & is_2 \end{pmatrix} \mid s_1, s_2 \in \mathbb{R},\ z \in \mathbb{C} \right\}.$$

Using the representation theory for SU(2) and the epimorphism

$$\mathbb{T} \times \mathsf{SU}(2) \ni (z, g) \mapsto zg \in \mathsf{U}(2),$$

we can determine all the representations $\pi^{(\ell,m)}$ of U(2),

$$\pi^{(\ell,m)} : \mathsf{U}(2) \to \mathsf{U}(\ell + 1), \quad \ell \in \mathbb{N},\ m \in \mathbb{Z}.$$

If $\phi \in C^1\big(X; \mathrm{U}(2)\big)$, then

$$D_{\phi,\ell,m} := D_{\phi,\pi^{(\ell,m)}} = \operatorname*{s-lim}_{N\to\infty} \frac{1}{N}\Big[A, \big(U_{\phi,\pi^{(\ell,m)},j}\big)^N\Big]\big(U_{\phi,\pi^{(\ell,m)},j}\big)^{-N}$$

exists and is equal to $i\big(\mathrm{d}\pi^{(\ell,m)}\big)_{I_2}\big((P_\phi M_\phi)(\cdot)\big)$ with $I_2 := \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) = e_{\mathrm{U}(2)}$, and U_ϕ is mixing in the subspace

$$\bigoplus_{m\in\mathbb{Z}} \bigoplus_{\ell\in\mathbb{N}} \bigoplus_{j\in\{0,\dots,\ell\}} \ker(D_{\phi,\ell,m})^\perp \subset \mathcal{H}.$$

To say more on U_ϕ, we further assume that T_ϕ is uniquely ergodic. In this case, we have $D_{\phi,\ell,m} = i\big(\mathrm{d}\pi^{(\ell,m)}\big)_{I_2}(M_{\phi,\star})$ with

$$M_{\phi,\star} \in z\big(\mathfrak{u}(2)\big) = \big\{isI_2 \mid s \in \mathbb{R}\big\}$$

Therefore, $M_{\phi,\star} = is_\phi I_2$ for some $s_\phi \in \mathbb{R}$.

Using an explicit formula for $\pi^{(\ell,m)}$, we obtain

$$i\big(\mathrm{d}\pi^{(\ell,m)}\big)_{I_2}(M_{\phi,\star})_{j,k} = -s_\phi\,(2m-\ell)\,j\,!(\ell-j)!\,\delta_{j,k}, \quad j,k\in\{0,\dots,\ell\}.$$

(constant diagonal matrix)

Thus, if $s_\phi \neq 0$, U_ϕ is mixing in the subspace

$$\mathcal{H}_{\text{mix}} := \bigoplus_{m\in\mathbb{Z}} \bigoplus_{\ell\in\mathbb{N}\setminus\{2m\}} \bigoplus_{j\in\{0,\dots,\ell\}} \mathcal{H}_j^{(\pi^{(\ell,m)})} \subset \mathcal{H}.$$

Under an additional regularity assumption of Dini-type on $\mathscr{L}_Y(\pi^{(\ell,m)}\circ\phi)$, we obtain that U_ϕ has purely absolutely continuous spectrum in $\mathcal{H}_{\text{mix}}$.

Example

Using the isomorphism

$$\mathrm{SO}(3,\mathbb{R}) \times \mathbb{T} \simeq \mathrm{U}(2)$$

and the works of Eliasson and Hou on skew products on $\mathbb{T}^d \times \mathrm{SO}(3,\mathbb{R})$, *we can produce skew-products* T_ϕ *on* $\mathbb{T}^d \times \mathrm{U}(2)$ *satisfying the assumptions.*

Namely, skew-products T_ϕ *with* $\phi \in C^\infty\left(\mathbb{T}^d; \mathrm{U}(2)\right)$, T_ϕ *uniquely ergodic, and nonzero degree* $M_{\phi,\star} = i s_\phi I_2$.

Thank you !

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