Quantum time delay for unitary operators

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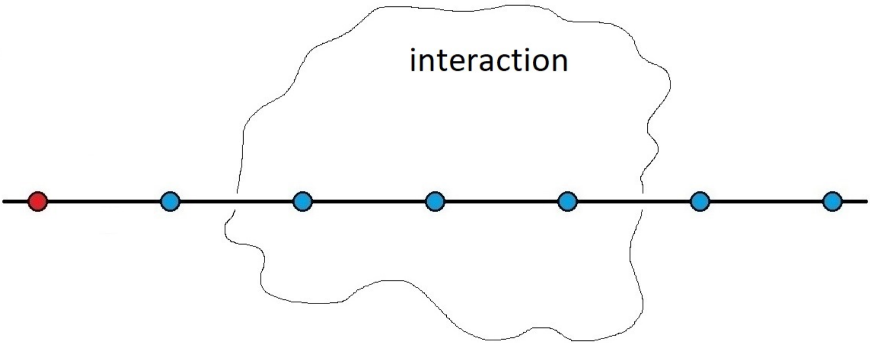
Table of Contents

Discrete-time unitary evolution without interaction



Discrete-time unitary evolution with interaction

Free propagator and position operators

- U_0, unitary operator with spectrum $\sigma(U_0) \subset \mathbb{S}^1$ and spectral measure E^{U_0} in a Hilbert space $\mathcal{H}_0$ (free propagator)

- $Q := (Q_1, \ldots, Q_d)$, family of mutually commuting self-adjoint operators in $\mathcal{H}_0$ (position operators)

- Evolution of U_0 along the flow generated by Q

$$U_0(x) := \mathrm{e}^{-ix\cdot Q}\, U_0\, \mathrm{e}^{ix\cdot Q}, \quad x \in \mathbb{R}^d.$$

Assumption (Regularity)

The map

$$\mathbb{R}^d \ni x \mapsto U_0(x) U_0^{-1} \in \mathscr{B}(\mathcal{H}_0)$$

is strongly differentiable on a suitable dense set $\mathscr{D} \subset \mathcal{H}_0$*, and the operators*

$$V_j \varphi := \text{s-} \frac{\mathrm{d}}{\mathrm{d}x_j}\Big|_{x=0} i U_0(x) U_0^{-1} \varphi, \quad j \in \{1, \ldots, d\},\ \varphi \in \mathscr{D},$$

are essentially self-adjoint (+ some technical conditions).

$V := (V_1, \ldots, V_d)$ is the velocity vector associated to U_0 and Q.

Assumption (Commutation)

$U_0(x)$ and $U_0(y)$ mutually commute for all $x, y \in \mathbb{R}^d$.

This implies that $U_0(x)$ and V_j mutually commute for all $j \in \{1, \ldots, d\}$ and $x, y \in \mathbb{R}^d$.

"Good" and "Bad" values in the spectrum of U_0 :

Definition (Critical values)

$\mathrm{e}^{i\lambda} \in \mathbb{S}^1$ is a regular value if there is $\delta > 0$ such that

$$\lim_{\varepsilon \searrow 0} \left\| \left(V^2 \left(V^2 + 1 \right)^{-1} + \varepsilon \right)^{-1} E^{U_0}(\lambda; \delta) \right\|_{\mathscr{B}(\mathcal{H}_0)} < \infty$$

where

$$E^{U_0}(\lambda; \delta) := E^{U_0}\left(\{ \mathrm{e}^{i\theta} \mid \theta \in (\lambda - \delta, \lambda + \delta) \} \right).$$

$\mathrm{e}^{i\lambda} \in \mathbb{S}^1$ is a critical value if it is not a regular value, and $\kappa(U_0)$ is the set of critical values of U_0.

In short, $\kappa(U_0)$ is the set of values in $\sigma(U_0)$ where $V = 0$.

Locally U_0-smooth operators

One can construct a self-adjoint operator A in $\mathcal{H}_0$ such that $[A, U_0]$ and $\big[A, [A, U_0]\big]$ are bounded in some suitable sense, and

$$U_0^{-1}[A, U_0] = \sum_{j=1}^{d} V_j^2 \big(V_j^2 + 1\big)^{-1} = \text{strictly positive on } \mathbb{S}^1 \setminus \kappa(U_0).$$

Using this positivity, one obtains:

Theorem (Locally U_0-smooth operators)

Let the assumptions be satisfied.

(a) *The spectrum of U_0 in $\sigma(U_0) \setminus \kappa(U_0)$ is purely absolutely continuous.*

(b) *Each operator $B \in \mathscr{B}\big(\mathcal{D}(\langle Q\rangle^{-s}), \mathcal{H}_0\big)$, with $s > 1/2$, is locally U_0-smooth on $\mathbb{S}^1 \setminus \kappa(U_0)$.*

(b) means that for any closed set $\Theta \subset \mathbb{S}^1 \setminus \kappa(U_0)$ there exists $c_\Theta \geq 0$ such that

$$\sum_{n \in \mathbb{Z}} \big\| B\, U_0^n E^{U_0}(\Theta)\varphi \big\|^2_{\mathcal{H}_0} \leq c_\Theta \|\varphi\|^2_{\mathcal{H}_0} \quad \text{for all } \varphi \in \mathcal{H}_0.$$

Dumb example

Take $Q=(Q_1,\dots,Q_d)$ and $P=(P_1,\dots,P_d)$ the position and momentum operators in $\mathcal H_0:=\mathsf L^2(\mathbb R^d)$, and $U_0:=\mathrm e^{-iP^2}$. Then,

$$U_0(x)=\mathrm e^{-ix\cdot Q}\,\mathrm e^{-iP^2}\,\mathrm e^{ix\cdot Q}=\mathrm e^{-i(P+x)^2},\quad x\in\mathbb R^d,$$

and $iU_0(x)U_0^{-1}=i\,\mathrm e^{-i(P+x)^2}\,\mathrm e^{iP^2}$ is strongly differentiable on $\mathscr D:=\mathscr S(\mathbb R^d)$ with $V=2P$. Thus, for $\mathrm e^{i\lambda}\in\mathbb S^1$ and $\delta>0$

$$\begin{aligned}&\lim_{\varepsilon\searrow0}\Big\|\big(V^2(V^2+1)^{-1}+\varepsilon\big)^{-1}E^{U_0}(\lambda;\delta)\Big\|_{\mathscr B(\mathcal H_0)}\\&=\lim_{\varepsilon\searrow0}\Big\|\big(4P^2(4P^2+1)^{-1}+\varepsilon\big)^{-1}E^{\mathrm e^{-iP^2}}(\lambda;\delta)\Big\|_{\mathscr B(\mathcal H_0)},\end{aligned}$$

so that $\kappa(U_0)=\{1\}$ and $\sigma(U_0)=\sigma_{\mathrm{ac}}(U_0)$.

Summation formula and Mackey imprimitivity theorem

For $t \geq 0$, let

$$\mathcal{D}_t := \big\{\varphi \in \mathcal{D}(\langle Q\rangle^t) \mid \varphi = \zeta(U_0)\varphi \text{ for some } \zeta \in C_c^\infty\big(\mathbb{S}^1 \setminus \kappa(U_0)\big) \\ + \text{ some technical condition}\big\}.$$

Theorem (Summation formula)

Let the assumptions be satisfied, and let $f \in \mathscr{S}(\mathbb{R}^d)$ be real, even and equal to 1 on a neighbourhood of $0 \in \mathbb{R}^d$. Then, we have for $\varphi \in \mathcal{D}_2$

$$\lim_{\nu \searrow 0} \frac{1}{2} \sum_{n \geq 0} \big\langle \varphi, \big(U_0^{-n} f(\nu Q) U_0^n - U_0^n f(\nu Q) U_0^{-n}\big)\varphi\big\rangle_{\mathcal{H}_0} = \big\langle \varphi, T_f\, \varphi\big\rangle_{\mathcal{H}_0},$$

with T_f a pseudo-differential operator (in Q and V) satisfying on $\mathcal{D}_1$

$$U_0^{-1}[T_f, U_0] = -1.$$

- For each $\nu > 0$,

$$\left|\sum_{n\geq 0}\left\langle\varphi,\left(U_0^{-n}f(\nu Q)U_0^n - U_0^n f(\nu Q)U_0^{-n}\right)\varphi\right\rangle_{\mathcal{H}_0}\right| < \infty$$

  because the operator $|f(\nu Q)|^{1/2}$ is locally U_0-smooth on $\mathbb{S}^1 \setminus \kappa(U_0)$ and φ has support outside $\kappa(U_0)$.

- The proof of the theorem consists in calculating the sum, and then doing $\lim_{\nu\searrow 0}$ (taking directly the limit inside the sum is not possible, and would give zero!).

- The proof involves commutator methods for families of operators, Trotter-Kato formula, domain arguments and a repeated use of Lebesgue's dominated convergence theorem...

- The relation

$$U_0^{-1}\,[T_f, U_0] = -1.$$

is the unitary analogue of the canonical time-energy commutation relation $[H, T] = i$. So, T_f is a time operator for U_0, and should be equal in some sense to the operator $-U_0\,\frac{\mathrm{d}}{\mathrm{d}U_0}$.

Using Mackey imprimitivity theorem (generalisation of Stone-von Neumann theorem), one shows under certain conditions that T_f acts as the differential operator $-z\,\frac{\mathrm{d}}{\mathrm{d}z}$ $(z \in \mathbb{S}^1)$ in a Hilbert space isomorphic to $\mathcal{H}_0$.

- The l.h.s. of the formula is the difference of times spent by the state $U_0^n \varphi$ in the future (first term) and in the past (second term) in the region defined by the localisation operator $f(\nu Q)$.

  So, the formula says that this difference of times converges as $\nu \searrow 0$ to the expectation value in φ of the time operator T_f.

Dumb example (Continued)

If $\mathcal{H}_0 = \mathsf{L}^2(\mathbb{R}^d)$ *and* $U_0 = \mathrm{e}^{-iP^2}$*, then*

$$\mathcal{D}_1 = \big\{\varphi \in \mathcal{D}(\langle Q\rangle) \mid \varphi = \zeta(\mathrm{e}^{-iP^2})\varphi \text{ for some } \zeta \in C_{\mathrm{c}}^{\infty}(\mathbb{S}^1 \setminus \{1\}) \\ + \text{ some technical condition}\big\}.$$

is dense in $\mathcal{H}_0$*. Furthermore, if* f *radial, then*

$$T_f = -\tfrac{1}{4}\left(Q \cdot \tfrac{P}{P^2} + \tfrac{P}{P^2} \cdot Q\right) \quad \text{on} \quad \mathcal{D}_1.$$

Thus, T *is symmetric on* $\mathcal{D}_1$ *and acts as* $-z\,\frac{\mathrm{d}}{\mathrm{d}z}$ ($z \in \mathbb{S}^1$) *in the spectral representation of* U_0*.*

Quantum time delay

- U, unitary operator in a Hilbert space $\mathcal{H}$ (full propagator)
- $J \in \mathscr{B}(\mathcal{H}_0, \mathcal{H})$, identification operator

Assumption (Wave operators)

The wave operators

$$W_\pm := \operatorname*{s-lim}_{n \to \pm\infty} U^{-n} J U_0^n P_{\mathrm{ac}}(U_0)$$

exist and are partial isometries with initial subspaces $\mathcal{H}_0^\pm \subset \mathcal{H}_0$ and final subspaces $\mathcal{H}_{\mathrm{ac}}(U)$.

Thus, the scattering operator

$$S := W_+^* W_- : \mathcal{H}_0^- \to \mathcal{H}_0^+$$

is a well-defined unitary operator commuting with U_0.

For $r > 0$, $f \in \mathscr{S}(\mathbb{R}^d)$ non-negative and equal to 1 on a neighbourhood Σ of $0 \in \mathbb{R}^d$, and suitable $\varphi \in \mathcal{H}_0$, we define

$$T_r^0(\varphi) := \sum_{n\in\mathbb{Z}} \big\langle U_0^n\varphi, f(Q/r)U_0^n\varphi\big\rangle_{\mathcal{H}_0}$$

$$T_{r,1}(\varphi) := \sum_{n\in\mathbb{Z}} \big\langle J^*U^nW_-\varphi, f(Q/r)J^*U^nW_-\varphi\big\rangle_{\mathcal{H}_0}$$

$$T_2(\varphi) := \sum_{n\in\mathbb{Z}} \big\langle U^nW_-\varphi, (1-JJ^*)U^nW_-\varphi\big\rangle_{\mathcal{H}_0}.$$

If $\|\varphi\|_{\mathcal{H}_0} = 1$, then

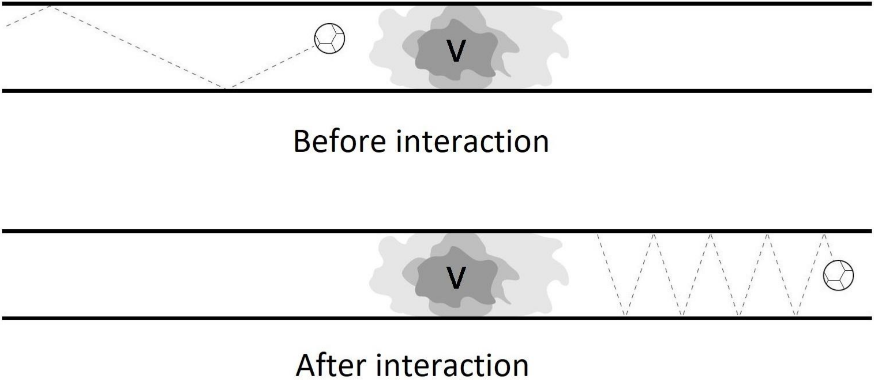
- $T_r^0(\varphi)$, time spent by $U_0^n\varphi$ inside $E^Q(r\Sigma)\mathcal{H}_0$,
- $T_{r,1}(\varphi)$, time spent by $U^nW_-\varphi$ inside $E^Q(r\Sigma)\mathcal{H}_0$ after being injected in $\mathcal{H}_0$ by J^*,
- $T_2(\varphi)$, time spent by $U^nW_-\varphi$ inside $(1-JJ^*)\mathcal{H}$.

The symmetrised time delay of the scattering system (U_0, U, J, φ) in the region defined by $f(Q/r)$ is

$$\tau_r^{\mathrm{sym}}(\varphi) := \big(T_{r,1}(\varphi) + T_2(\varphi)\big) - \tfrac{1}{2}\big(T_r^0(\varphi) + T_r^0(S\varphi)\big).$$

The non-symmetrised time delay of the scattering system (U_0, U, J, φ) in the region defined by $f(Q/r)$ is

$$\tau_r^{\mathrm{nsym}}(\varphi) := \big(T_{r,1}(\varphi) + T_2(\varphi)\big) - T_r^0(\varphi).$$



Classical scattering in a waveguide: The non-symmetrised time delay does not exist because the longitudinal velocities before and after interaction are not comparable.

Theorem (Symmetrised time delay)

Let the assumptions be satisfied. Let $f \in \mathscr{S}(\mathbb{R}^d)$ be non-negative, even and equal to 1 on a neighbourhood of $0 \in \mathbb{R}^d$. Let $\varphi \in \mathcal{H}_0^- \cap \mathcal{D}_2$ satisfy $S\varphi \in \mathcal{D}_2$ plus some technical assumption. Then,

$$\lim_{r\to\infty} \tau_r^{\mathrm{sym}}(\varphi) = \big\langle \varphi, S^*[T_f, S]\varphi \big\rangle_{\mathcal{H}_0}. \tag{$\bigstar$}$$

- The l.h.s. of ($\bigstar$) is the symmetrised time delay as $r \to \infty$, and the r.h.s. of ($\bigstar$) is the expectation value in φ of the time delay operator $S^*[T_f, S]$.
- When T_f acts as $-U_0 \frac{\mathrm{d}}{\mathrm{d}U_0}$, one gets an analogue of Eisenbud-Wigner formula for unitary scattering systems:

$$\lim_{r\to\infty} \tau_r^{\mathrm{sym}}(\varphi) = \left\langle \varphi, -S^* U_0 \frac{\mathrm{d}S}{\mathrm{d}U_0} \varphi \right\rangle_{\mathcal{H}_0}.$$

- The main ingredient of the proof is the summation formula.

Theorem (Non-symmetrised time delay)

If the assumptions of the previous theorem are satisfied and $[V^2, S] = 0$, then

$$\lim_{r\to\infty} \tau_r^{\mathrm{nsym}}(\varphi) = \lim_{r\to\infty} \tau_r^{\mathrm{sym}}(\varphi) = \big\langle \varphi, S^*[T_f, S]\varphi \big\rangle_{\mathcal{H}_0}.$$

The relation $[V^2, S] = 0$ means that the norm of the velocity is preserved by the scattering process. Thus, the symmetrisation is not needed because the norm of the velocities before and after interaction are comparable.

Gracias !

References

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