New formulae for the wave operators for a rank one interaction

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In $\mathcal{H} := \mathsf{L}^2(\mathbb{R})$, let $\mathsf{H}_0 \equiv \mathsf{X}$ be the operator of multiplication by the variable

$$(\mathsf{H}_0 f)(x) := x f(x), \qquad f \in \mathcal{D}(\mathsf{H}_0) \equiv \mathcal{H}_1 .$$

For $u \in \mathcal{H}$, we consider the rank one perturbation of H_0

$$\mathsf{H}_u f := \mathsf{H}_0 f + \langle u, f \rangle \, u.$$

- The wave operators $\Omega_\pm := \text{s-}\lim_{t \to \pm\infty} e^{itH_u} \, e^{-itH_0}$ exist and are asymptotically complete.
- The scattering operator $S := \Omega_+^* \Omega_-$ is unitary.

  ($S \equiv S(\mathsf{X})$ is an operator of multiplication by a function $\mathsf{S} : \mathbb{R} \to \mathbb{T}$)

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The vector u satisfies a weak regularity condition:

Assumption 2.1. *The function* $u \in \mathcal{H}$ *is Hölder continuous with exponent* $\alpha > 0$*. Furthermore, if* $x_0 \in \mathbb{R}$ *satisfies* $u(x_0) = 0$ *and* $1 - \int_{\mathbb{R}} dy \, \frac{|u(y)|^2}{x_0 - y} = 0$*, then there exists an exponent* $\alpha' > 1/2$ *such that*

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- If $u \in \mathcal{H}$ is Hölder continuous with exponent $\alpha > 1/2$, then the assumption holds.
- If u satisfies the assumption, then u is bounded and satisfies $\lim_{|x| \to \infty} u(x) = 0$.
- If u satisfies the assumption, then H_u has at most a finite number of eigenvalues of multiplicity one (*cf.* [Alsholm80]).

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$\Longrightarrow$ The domain $\mathcal{D}_\pm$ of $[1 - I_\pm(X)]^{-1}$ in $\mathcal{H}$ is dense.

Grubb/Pearson have proved on $\mathcal{D}_+$ that

$$\begin{aligned}\Omega_- &= 1 - 2\pi i\, u(X)\, \chi_{(-\infty,0)}(D)\, \overline{u}(X)\, [1 - I_+(X)]^{-1},\\ S(X) - 1 &= -2\pi i\, |u(X)|^2\, [1 - I_+(X)]^{-1},\end{aligned}$$

where $D := -i\,\partial_x$ and $\chi_{(-\infty,0)}$ is the characteristic function for $(-\infty, 0)$.

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So, if we set

$$K := -2\pi i \left[u(X), \chi_{(-\infty,0)}(D)\right] \overline{u}(X)\, [1 - I_+(X)]^{-1},$$

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(This is the starting point of our study)

Let $\mathscr{U} : \mathcal{H} \to \mathsf{L}^2(\mathbb{R}_+; \mathbb{C}^2)$ be the unitary operator given by

$$(\mathscr{U}\,\mathsf{f}) := \sqrt{2}\begin{pmatrix} \mathsf{f}_{\mathrm{e}} \\ \mathsf{f}_{\mathrm{o}} \end{pmatrix} \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} \mathsf{f}(\,\cdot\,) + \mathsf{f}(-\,\cdot\,) \\ \mathsf{f}(\,\cdot\,) - \mathsf{f}(-\,\cdot\,) \end{pmatrix}.$$

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If m is a function on $\mathbb{R}$ and A the generator of dilations in $\mathcal{H}$, then

$$\mathscr{U}\,\mathsf{m}(\mathsf{X})\mathscr{U}^* = \begin{pmatrix} \mathsf{m}_\mathrm{e}(\mathsf{X}_+) & \mathsf{m}_\mathrm{o}(\mathsf{X}_+) \\ \mathsf{m}_\mathrm{o}(\mathsf{X}_+) & \mathsf{m}_\mathrm{e}(\mathsf{X}_+) \end{pmatrix} \qquad \text{and} \qquad \mathscr{U} A \mathscr{U}^* = \begin{pmatrix} A_+ & 0 \\ 0 & A_+ \end{pmatrix},$$

where X_+ is the operator of multiplication by the variable in $\mathsf{L}^2(\mathbb{R}_+)$, and A_+ the generator of dilations in $\mathsf{L}^2(\mathbb{R}_+)$.

Our new formula for Ω_- is the following.

Theorem 3.1. *Let* u *satisfy Assumption 2.1. Then, one has*

$$\mathscr{U}\,\Omega_-\mathscr{U}^*$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 & \tanh(\pi \mathsf{A}_+) - \frac{i}{\cosh(\pi \mathsf{A}_+)} \\ \tanh(\pi \mathsf{A}_+) + \frac{i}{\cosh(\pi \mathsf{A}_+)} & -1 \end{pmatrix}\begin{pmatrix} \mathsf{S}_{\mathrm{e}}(\mathsf{X}_+) - 1 & \mathsf{S}_{\mathrm{o}}(\mathsf{X}_+) \\ \mathsf{S}_{\mathrm{o}}(\mathsf{X}_+) & \mathsf{S}_{\mathrm{e}}(\mathsf{X}_+) - 1 \end{pmatrix} + \mathsf{K},$$

where K *is a compact operator in* $\mathscr{H}$.

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where K *is a compact operator in* $\mathscr{H}$*.*

Idea of the proof. The general expression follows from the fomula

$$\Omega_- - 1 = \chi_{(-\infty,0)}(D)\{S(X) - 1\} + K.$$

and the computation of $\mathscr{U}\chi_{(-\infty,0)}(D)\mathscr{U}^*$ using Hilbert transform.

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and the computation of $\mathscr{U}\chi_{(-\infty,0)}(D)\mathscr{U}^*$ using Hilbert transform. The fact that K is compact follows from the assumption on $\mathfrak{u}$ and some results on singular integrals. $\square$

4 Topological version of Levinson′s theorem

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Corollary 4.1. *Let* $\mathfrak{u}$ *satisfy Assumption 2.1. Then the following equality holds*

$$\omega(S) = -\textit{ number of eigenvalues of } H_{\mathfrak{u}}\,,$$

where $\omega(S)$ *is the winding number of the continuous map* $S : \mathbb{R} \to \mathbb{T}$.

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1. In $\mathscr{H}$, Ω_- is an element of the type $\varphi(\mathsf{A}_+)\psi(\mathsf{X}_+) + \mathsf{K}$, with $\varphi \in \mathsf{C}\big(\overline{\mathbb{R}}; \mathsf{M}_2(\mathbb{C})\big)$, $\psi \in \mathsf{C}\big(\overline{\mathbb{R}_+}; \mathsf{M}_2(\mathbb{C})\big)$, and K compact.

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$$\text{index}(\Omega_-) = -\text{ number of eigenvalues of } H_u\,.$$

3. Georgescu/Iftimovici have proved that the image $q(\Omega_-)$ of Ω_- in the Calkin algebra $\mathscr{B}(\mathscr{H})/\mathcal{K}(\mathscr{H})$ is isomorphic to a (invertible)matrix-valued function in $C\big(\mathbb{S}; M_2(\mathbb{C})\big)$.

4. Viewing $\mathsf{q}(\Omega_-)$ as an element of $\mathsf{C}\big(\mathbb{S}; M_2(\mathbb{C})\big)$, we can take pointwise its determinant to obtain a non-vanishing function over the circle. The winding number of that function satisfies

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(The relation $\omega\big(\mathsf{q}(\Omega_-)\big) = \text{index}(\Omega_-)$ has been proved by Kellendonk/Richard using K-theory.)

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Davies has considered the perturbed Hamiltonian

$$H_{u_\lambda} := H_0 + \langle u_\lambda, \cdot \rangle \, u_\lambda,$$

where

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(Two simulations: one for $\lambda < \sqrt{2/\pi}$, and one for $\lambda = \sqrt{2/\pi}$.)

7 Some references

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