A new formula relating localisation operators to time operators

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Roughly, the operator T is called a **time operator for H** if it satisfies the canonical commutation relation

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or the relation

$$\mathsf{T}\,\mathrm{e}^{-\mathsf{i}t\mathsf{H}} = \mathrm{e}^{-\mathsf{i}t\mathsf{H}}(\mathsf{T} + \mathsf{t})\,.$$

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But, in most cases, the pair $\{\mathsf{H}, \mathsf{T}\}$ is given from the beginning !

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... this is the subject of the talk.

2 Guiding example

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Consider in $\mathsf{L}^2(\mathbb{R}^d)$ a dispersive operator $H := h(P)$ with $h : \mathbb{R}^d \to \mathbb{R}$ and $P \equiv (-i\partial_1, \dots, -i\partial_d)$.

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Then one has for suitable $\varphi \in L^2(\mathbb{R}^d)$:

$$\lim_{r\to\infty} \frac{1}{2} \int_0^\infty dt \left\langle \varphi, \left[e^{-itH} f(Q/r)\, e^{itH} - e^{itH} f(Q/r)\, e^{-itH} \right] \varphi \right\rangle = \left\langle \varphi, i \tfrac{d}{dH}\, \varphi \right\rangle,$$

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where $\frac{d}{dH}$ acts as $\frac{d}{d\lambda}$ in the spectral representation of H, *i.e.*

$$\big[i \tfrac{d}{dH}, H \big] = i \, .$$

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The formula can be extended to the case of an abstract pair of operator H and position operators Φ acting in a Hilbert space $\mathcal{H}$, if H and Φ satisfy two appropriate commutation relations.

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Two assumptions:

(A) The family of operators $H(x) := e^{-ix\cdot\Phi}\, H\, e^{ix\cdot\Phi}$, $x \in \mathbb{R}^d$, mutually commute.

(B) For some $\omega \in \mathbb{C} \setminus \mathbb{R}$ the map

$$\mathbb{R}^d \ni x \mapsto \big[H(x) - \omega\big]^{-1} \in \mathscr{B}(\mathcal{H})$$

is 3-times strongly differentiable.

4 Main theorem

Under Assumptions (A) and (B), we have:

Theorem 4.1. *Let* f *be a Schwartz function on* $\mathbb{R}^d$ *such that* $f = 1$ *on a neighbourhood of* 0 *and* $f(x) = f(-x)$ *for each* $x \in \mathbb{R}^d$*. Then, for each* φ *in some suitable subset of* $\mathcal{H}$ *one has*

$$\lim_{r\to\infty} \frac{1}{2}\int_0^\infty \mathrm{d}t \left\langle \varphi, \left[e^{-itH} f(\Phi/r)\, e^{itH} - e^{itH} f(\Phi/r)\, e^{-itH} \right] \varphi \right\rangle = \langle \varphi, T_f \varphi \rangle, \tag{T-Op}$$

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where the operator T_f *acts, in an appropriate sense, as* $i\frac{\mathrm{d}}{\mathrm{d}\lambda}$ *in the spectral representation of* H.

(T_f is a time operator for H.)

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Let H_j' be the self-adjoint operator associated with $i[H, \Phi_j]$ and $H' := \big(H_1', \dots, H_d'\big)$. Then T_f is formally given by

$$T_f = -\tfrac{1}{2}\big(\Phi \cdot R_f'(H') + R_f'(H') \cdot \Phi\big),$$

where

$$R_f : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}, \qquad R_f(x) := \int_0^{+\infty} \frac{d\mu}{\mu}\big[f(\mu x) - \chi_{[0,1]}(\mu)\big].$$

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When f is radial, $R_f'(x) = -x^{-2}x$ and

$$T_f \equiv T = \tfrac{1}{2}\big(\Phi \cdot \tfrac{H'}{(H')^2} + \tfrac{H'}{(H')^2} \cdot \Phi\big).$$

Thus

$$
\begin{aligned}
[T, H] &= \frac{1}{2}\sum_j \left\{[\Phi_j, H]\frac{H'_j}{(H')^2} + \frac{H'_j}{(H')^2}[\Phi_j, H]\right\} \\
&= \frac{i}{2}\sum_j \left\{H'_j\frac{H'_j}{(H')^2} + \frac{H'_j}{(H')^2}H'_j\right\} \\
&= i.
\end{aligned}
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This suggests the definition:

Definition 4.2. *A number $\lambda \in \mathbb{R}$ is a **critical value of** H if*

$$\lim_{\varepsilon \searrow 0} \left\| \left[(H')^2 + \varepsilon \right]^{-1} E^H \big((\lambda - \delta, \lambda + \delta) \big) \right\| = +\infty.$$

for all $\delta > 0$. We denote by $\kappa(H)$ the set of critical values of H.

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So, the vectors $\varphi \in \mathcal{H}$ in the theorem are chosen such that $\varphi = E^H(J)\varphi$ for some set $J \subset \mathbb{R} \setminus \kappa(H)$.

This definition extends the usual definition of critical values when $H = h(P)$ and $\Phi = Q$:

$$\kappa_h := \left\{\lambda \in \mathbb{R} \mid \exists x \in \mathbb{R}^d \text{ such that } h(x) = \lambda \text{ and } h'(x) = 0\right\}.$$

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(proof by commutators methods)

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Thus

$$\tau(\varphi) = -\langle \varphi, S^*[T_f, S]\varphi \rangle,$$

and

$$\tau(\varphi) = \left\langle \varphi, -iS^* \tfrac{dS}{dH}\, \varphi \right\rangle,$$

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(Abstract Eisenbud-Wigner Formula)

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$$= \int_{\mathbb{R}^d} \underline{d}x \, (\mathscr{F} f)(x) \left\langle \varphi, \left[e^{i\frac{x}{r}\cdot\Phi} e^{it[H(\frac{x}{r})-H]} - e^{it[H(-\frac{x}{r})-H]} e^{i\frac{x}{r}\cdot\Phi} \right] \varphi \right\rangle$$

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$$\begin{aligned}
&\Big\langle \varphi, \big[\, e^{itH}\, f(\Phi/r)\, e^{-itH} - e^{-itH}\, f(\Phi/r)\, e^{itH} \,\big]\varphi \Big\rangle \\
&= \int_{\mathbb{R}^d} \underline{d}x\, (\mathscr{F} f)(x) \Big\langle \varphi, \big[\, e^{itH}\, e^{i\frac{x}{r}\cdot\Phi}\, e^{-itH} - e^{-itH}\, e^{i\frac{x}{r}\cdot\Phi}\, e^{itH} \,\big]\varphi \Big\rangle \\
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&= \int_{\mathbb{R}^d} \underline{d}x\, (\mathscr{F} f)(x) \Big\langle \varphi, \Big[\big(e^{i\frac{x}{r}\cdot\Phi} - 1\big)\, e^{it[H(\frac{x}{r})-H]} - e^{it[H(-\frac{x}{r})-H]}\, \big(e^{i\frac{x}{r}\cdot\Phi} - 1\big)\Big]\varphi \Big\rangle \\
&\quad + \underbrace{\int_{\mathbb{R}^d} \underline{d}x\, (\mathscr{F} f)(x) \Big\langle \varphi, \big[\, e^{it[H(\frac{x}{r})-H]} - e^{it[H(-\frac{x}{r})-H]} \,\big]\varphi \Big\rangle}_{=0 \text{ because } f(x)=f(-x)}
\end{aligned}$$

So, after the changes of variables $\mu := t/r$, $\nu := 1/r$, one gets

$$\lim_{r\to\infty} \frac{1}{2} \int_0^\infty dt \, \big\langle \varphi, \big[e^{-itH} f(\Phi/r) \, e^{itH} - e^{itH} f(\Phi/r) \, e^{-itH} \big] \varphi \big\rangle$$

$$= -\frac{1}{2} \lim_{\nu \searrow 0} \int_0^\infty d\mu \int_{\mathbb{R}^d} \underline{d}x \, (\mathscr{F} f)(x) \Big\langle \varphi, \Big\{ \frac{1}{\nu} \Big(e^{i\nu x \cdot \Phi} - 1 \Big) \, e^{i \frac{\mu}{\nu} [H(\nu x) - H]} - e^{i \frac{\mu}{\nu} [H(-\nu x) - H]} \, \frac{1}{\nu} \Big(e^{i\nu x \cdot \Phi} - 1 \Big) \Big\} \varphi \Big\rangle.$$

So, after the changes of variables $\mu := t/r$, $\nu := 1/r$, one gets

$$\lim_{r\to\infty} \frac{1}{2}\int_0^\infty dt \left\langle \varphi, \left[e^{-itH} f(\Phi/r)\, e^{itH} - e^{itH} f(\Phi/r)\, e^{-itH} \right] \varphi \right\rangle$$

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The rest of the proof consist in justifying the interchange of the limit with the integrals...

6 Examples

Example 6.1 (H' constant). *Suppose that* $H' = \nu \in \mathbb{R}^d \setminus \{0\}$. *Then*

- $H(x) = H + x \cdot \nu$ *is* C^∞ *and mutually commuting,*
- $\kappa(H) = \varnothing$,
- $\sigma(H) = \sigma_{\mathrm{ac}}(H) = \mathbb{R}$,
- $T_f = -R_f'(\nu) \cdot \Phi$.

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Friedrichs-type Hamiltonians $H := \nu \cdot P + V(Q)$ *and Stark Hamiltonians* $H := P^2 + \nu \cdot Q$ *with position operators* $\Phi := Q$ *and* $\Phi := P$ *in* $L^2(\mathbb{R}^d)$ *fit into this construction.*

Example 6.2 ($H' = H$)**.** *Suppose, with* $d = 1$*, that* $H(x) = e^x H$ *(*C^∞ *and mutually commuting). Then*

- $H' = H$,
- $\kappa(H) = \{0\}$,
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In the Hilbert space $\mathcal{H} := \ell^2(\mathbb{N})$*, the Jacobi operator*

$$(H\varphi)(n) := (n-1)\varphi(n-1) + (2n-1)\varphi(n) + n\varphi(n+1),$$

and

$$(\Phi\varphi)(n) := -\tfrac{i}{2}\big\{(n-1)\varphi(n-1) - n\varphi(n+1)\big\}.$$

fit into this construction.

Example 6.3 (Dirac)**.** *Consider in* $\mathcal{H} := \mathsf{L}^2(\mathbb{R}^3;\mathbb{C}^4)$ *the Dirac operator*

$$\mathsf{H} := \alpha \cdot \mathsf{P} + \beta m$$

and take the Wigner-Newton position operator

$$\Phi := \mathscr{U}_{\mathrm{FW}}^{-1}\mathsf{Q}\mathscr{U}_{\mathrm{FW}} \qquad (\mathscr{U}_{\mathrm{FW}} = \text{Foldy-Wouthuysen transformation}).$$

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Then $H(x) = \sqrt{\frac{(P+x)^2+m^2}{P^2+m^2}}\,H$ *is* C^∞ *and mutually commuting, and*

- $H_j' = P_j H^{-1}$,
- $\kappa(H) = \{\pm m\}$,
- $\sigma(H) = \sigma_{\mathrm{ac}}(H) = (-\infty, -m] \cup [m, \infty)$,
- $T = \frac{1}{2}\big\{\Phi \cdot PP^{-2}H + PP^{-2}H \cdot \Phi\big\}$ *when* f *is radial.*

Example 6.4 (Convolutions). *Let* G *be a locally compact group with a left Haar measure* ρ. *In* $\mathcal{H} := \mathsf{L}^2(\mathsf{G}, \mathrm{d}\rho)$ *we consider*

$$H_\mu \varphi := \mu * \varphi \qquad \text{for } \mu = \text{bounded Radon measure on } \mathsf{G},$$

and take $\Phi \equiv (\Phi_1, \dots, \Phi_d)$ *with* Φ_j *continuous group morphisms from* G *to* $\mathbb{R}$.

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Suppose that μ *has compact support and is central,* i.e. $\mu(h^{-1}Eh) = \mu(E)$ *for each* $h \in \mathsf{G}$ *and each Borel subset* E *of* G*.*

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Then

$$H_\mu(x)\varphi = \int_G d\mu(h)\, e^{-ix\cdot\Phi(h)}\, \varphi(h^{-1}\,\cdot\,)$$

is C^∞ *and mutually commuting.*

Example 6.5 ($\mathsf{H} = \mathsf{h}(\mathsf{P})$)**.** *Consider in* $\mathcal{H} := \mathsf{L}^2(\mathbb{R}^\mathrm{d})$ *the operator* $\mathsf{H} := \mathsf{h}(\mathsf{P})$*, where* $\mathsf{h} \in \mathsf{C}^3(\mathbb{R}^\mathrm{d};\mathbb{R})$ *satisfies some hypoelliptic decay assumptions, and take* $\Phi := \mathsf{Q}$*.*

Example 6.5 $(\mathsf{H} = \mathsf{h}(\mathsf{P}))$**.** *Consider in* $\mathcal{H} := \mathsf{L}^2(\mathbb{R}^d)$ *the operator* $\mathsf{H} := \mathsf{h}(\mathsf{P})$, *where* $\mathsf{h} \in \mathsf{C}^3(\mathbb{R}^d;\mathbb{R})$ *satisfies some hypoelliptic decay assumptions, and take* $\Phi := \mathsf{Q}$.

Then $\mathsf{H}(\mathsf{x}) = \mathsf{h}(\mathsf{P} + \mathsf{x})$ *is* C^3 *and mutually commuting.*

7 Some references

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