Asymptotics near $\pm \mathfrak{m}$ of the SSF for Dirac operators with non-constant magnetic fields

Rafael Tiedra
(Pontificia Universidad Católica de Chile)

Grenoble, February 2011

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Free Dirac Hamiltonian H_m acting in $\mathcal{H} := L^2(\mathbb{R}^3; \mathbb{C}^4)$ unitarily equivalent to the operator $h(P) \oplus -h(P)$, where

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(We study the spectral properties of Dirac operators with non-constant magnetic field near $\pm m$.)

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Variable magnetic field of constant direction

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One can choose an associated vector potential

$$\vec{a}(x_1, x_2, x_3) = \big(a_1(x_1, x_2), a_2(x_1, x_2), 0\big)$$

with $\vec{a} \in L^\infty_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^3)$.

Under these assumptions, the magnetic Dirac operator

$$H_0 := \alpha_1(P_1 - a_1) + \alpha_2(P_2 - a_2) + \alpha_3 P_3 + \beta m$$

is essentially selfadjoint on $C_0^\infty(\mathbb{R}^3; \mathbb{C}^4)$.

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We also have in $L^2(\mathbb{R}^3) \simeq L^2(\mathbb{R}^2) \otimes L^2(\mathbb{R})$ the decomposition

$$H_0^2 = \begin{pmatrix} H_\perp^- \otimes 1 + 1 \otimes (P_3^2 + m^2) & 0 & 0 & 0 \\ 0 & H_\perp^+ \otimes 1 + 1 \otimes (P_3^2 + m^2) & 0 & 0 \\ 0 & 0 & H_\perp^- \otimes 1 + 1 \otimes (P_3^2 + m^2) & 0 \\ 0 & 0 & 0 & H_\perp^+ \otimes 1 + 1 \otimes (P_3^2 + m^2), \end{pmatrix}$$

where $H_\perp^\pm$ are the components of the 2D Pauli operator

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The spectrum of H_0 is

$$\sigma(H_0) = \sigma_{\rm ac}(H_0) = (-\infty, -\mu_0] \cup [\mu_0, \infty), \qquad \mu_0 := \sqrt{\inf \sigma(H_\perp) + m^2}\,.$$

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Assumption 3.1. *The function* $\mathsf{V} \in \mathsf{C}\big(\mathbb{R}^3; \mathscr{B}_{\mathsf{h}}(\mathbb{C}^4)\big)$ *satisfies*

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The perturbed Hamiltonian

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is selfadjoint on $\mathcal{D}(\mathsf{H}_+) = \mathcal{D}(\mathsf{H}_0)$.

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(All what follows is also done for $H_- := H_0 - V$.)

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A sufficient condition on V:

(i) $[V, \alpha_1] = [V, \alpha_2] = 0$,

(ii) $|(\partial_\ell V_{jk})(x)| \leq \text{Const.} \langle x \rangle^{-\rho}$, $\rho > 3$, for each $j, k, \ell \in \{1, \ldots, 4\}$,

(iii) $(\partial_\ell \partial_3 V_{jk}) \in L^\infty(\mathbb{R}^3)$ for each $j, k, \ell \in \{1, \ldots, 4\}$.

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(Conditions (i) and (iii) are not necessary if V is scalar.)

There exists a unique function $\xi(\,\cdot\,; H_+, H_0) \in L^1\big(\mathbb{R}; (1+|\lambda|)^{-4} d\lambda\big)$ such that the Lifshits-Krein trace formula holds:

$$\operatorname{Tr}\big[f(H_+) - f(H_0)\big] = \int_{\mathbb{R}} d\lambda\, f'(\lambda)\, \xi(\lambda; H_+, H_0), \quad f \in C_0^\infty(\mathbb{R}).$$

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- For $\lambda_1, \lambda_2 \in (-m, m) \setminus \sigma(H_+)$ with $\lambda_1 < \lambda_2$, we have

$$\xi(\lambda_1; H_+, H_0) - \xi(\lambda_2; H_+, H_0) = \mathrm{rank}\, E^{H_+}\big([\lambda_1, \lambda_2)\big).$$

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- We have the Birman-Krein formula for almost every $\lambda \in \sigma_{ac}(H_0)$

$$\det S(\lambda; H_+, H_0) = e^{-2\pi i \xi(\lambda; H_+, H_0)}.$$

5 Magnetic field

As in [Raikov09], we assume that $\mathsf{b} = \mathsf{b}_0 + \widetilde{\mathsf{b}}$, where $\mathsf{b}_0 > 0$ is constant and $\widetilde{\mathsf{b}} : \mathbb{R}^2 \to \mathbb{R}$ is such that

$$\Delta \widetilde{\varphi} = \widetilde{\mathsf{b}}$$

admits a solution $\widetilde{\varphi} \in \mathsf{BC}^2(\mathbb{R}^2; \mathbb{R})$.

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Setting $\varphi_0(x_\perp) := \frac{1}{4} b_0 |x_\perp|^2$ and $\varphi := \varphi_0 + \widetilde{\varphi}$, we obtain that $\Delta\varphi_0 = b_0$, $\Delta\varphi = b$, and

$$a_1 := -\partial_2 \varphi, \qquad a_2 := \partial_1 \varphi,$$

gives a vector potential $\vec{a} \equiv (a_1, a_2, 0)$ for $\vec{B} \equiv (0, 0, b)$.

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(Changing, if necessary, the gauge, we assume that $\vec{a}$ is of this form.)

Remark 5.1. *For each* $x_\perp \in \mathbb{R}^2$, *the magnetic field* b *satisfies*

$$\lim_{r\to\infty} r^{-2} \int_{x_\perp + (-r/2, r/2)^2} d\xi \, b(\xi) = b_0 .$$

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(b_0 can be interpreted as the mean value of b.)

Example 5.2.

$$\widetilde{b}(x_\perp) := \int_{\mathbb{R}^2} d\eta(\xi)\ e^{i\xi\cdot x_\perp}, \quad x_\perp \in \mathbb{R}^2,$$

where η *is a complex measure such that*

- $|\eta|(\mathbb{R}^2) < \infty$,
- $\eta(\mathcal{B}) = \overline{\eta(-\mathcal{B})}$ *for each Borel set* $\mathcal{B} \subset \mathbb{R}^2$ *(*$\widetilde{b}$ *is real),*
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Then

$$\widetilde{\varphi}(x_\perp) := -\int_{\mathbb{R}^2} d\eta(\xi)\, |\xi|^{-2}\, e^{i\xi\cdot x_\perp}, \quad x_\perp \in \mathbb{R}^2,$$

is a suitable solution for $\Delta\widetilde{\varphi} = \widetilde{b}$.

Under these assumptions, it is known that:

- $0 = \inf \sigma(H_\perp)$ is an isolated eigenvalue of infinite multiplicity,
- there exists a constant $\zeta > 0$ such that $(0, \zeta) \subset \mathbb{R} \setminus \sigma(H_\perp)$.

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In particular, we have that $\mu_0 = m$, and thus

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It can be shown that

(a) $\xi(\,\cdot\,; H_+, H_0)$ is bounded on each compact subset of $(-\sqrt{m^2+\zeta}, \sqrt{m^2+\zeta}) \setminus \{\pm m\}$,

(b) $\xi(\,\cdot\,; H_+, H_0)$ is continuous on $(-\sqrt{m^2+\zeta}, \sqrt{m^2+\zeta}) \setminus \big(\{\pm m\} \cup \sigma_{\mathrm{p}}(H_+)\big)$.

6 Main theorems

The asymptotic behaviour of $\xi(\lambda; H_+, H_0)$ as $\lambda \to \pm m$, with $|\lambda| < m$, is given in terms of the Berezin-Toeplitz type operator

$$\omega_-(\lambda) := -\frac{1}{2}\left(\frac{m-\lambda}{m+\lambda}\right)^{1/2} pW_-p,$$

where p is the orthogonal projection onto $\ker\left(H_\perp^-\right)$, and

$$W_-(x_\perp) := \int_{\mathbb{R}} dx_3\, V_{33}(x_\perp, x_3), \quad x_\perp \in \mathbb{R}^2,$$

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(The operator $\omega_-(\lambda)$ is negative and belongs to $S_1[L^2(\mathbb{R}^2)]$.)

Given $\mathsf{T} = \mathsf{T}^*$ a compact operator in a separable Hilbert space $\mathcal{G}$, let

$$n_{\pm}(s; \mathsf{T}) := \operatorname{rank} \mathsf{E}^{\pm \mathsf{T}}\big((s, \infty)\big), \qquad s > 0.$$

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Theorem 6.1. *Let* V *satisfy the assumptions. Then one has for each* $\varepsilon \in (0,1)$

$$\mathcal{O}(1) \le \xi(\lambda; H_+, H_0) \le \mathcal{O}(1) \quad as \quad \lambda \nearrow m,$$

and

$$n_-\big(1+\varepsilon; \omega_-(\lambda)\big) + \mathcal{O}(1) \le \xi(\lambda; H_+, H_0) \le n_-\big(1-\varepsilon; \omega_-(\lambda)\big) + \mathcal{O}(1)$$

as $\lambda \searrow -m$.

The asymptotic behaviour of $\xi(\lambda; H_+, H_0)$ as $\lambda \to \pm m$, with $|\lambda| > m$, is given in terms of a Berezin-Toeplitz type operator

$$\Omega(\lambda) := \frac{1}{2\sqrt{\lambda^2 - m^2}}\, p M_\lambda p,$$

where $x_\perp \mapsto M_\lambda(x_\perp)$ is a positive 8×8 matrix-valued function defined in terms of $W_\pm$.

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The operator $\Omega(\lambda)$ is positive and satisfies

$$\|\Omega(\lambda)\|_1 \leq \left(\tfrac{\lambda+m}{\lambda-m}\right)^{1/2} \left\|pW_+p\right\|_1 + \left(\tfrac{\lambda-m}{\lambda+m}\right)^{1/2} \left\|pW_-p\right\|_1.$$

Theorem 6.2. *Let* V *satisfy the assumptions. Then one has for each* $\varepsilon \in (0, 1)$

$$\begin{aligned} &\pi^{-1}\,\mathrm{Tr}\arctan\left[(1+\varepsilon)^{-1}\Omega(\lambda)\right] + \mathcal{O}(1) \\ &\quad \le \xi(\lambda; H_+, H_0) \\ &\qquad \le \pi^{-1}\,\mathrm{Tr}\arctan\left[(1-\varepsilon)^{-1}\Omega(\lambda)\right] + \mathcal{O}(1) \end{aligned}$$

as $\lambda \to \pm m$, $|\lambda| > m$.

Remark 6.3. *One can explain the variation of* $\xi(\lambda; H_+, H_0)$ *under the transformation* $\lambda \mapsto -\lambda$ *via the charge conjugation symmetry*

$$C : \mathcal{H} \to \mathcal{H}, \quad \varphi \mapsto U_C \overline{\varphi},$$

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where $U_C := i\beta\alpha_2$. *Namely,*

$$\xi\big(\lambda; H(\vec{a}, V), H_0(\vec{a})\big) = -\xi\big(-\lambda; H(-\vec{a}, -U_C \overline{V} U_C^*), H_0(-\vec{a})\big).$$

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Roughly, the transformation $\lambda \mapsto -\lambda$ *leads to the changes:*

- $\xi \mapsto -\xi$,
- $\vec{a} \mapsto -\vec{a}$,
- $(V_{11}, V_{33}) \mapsto -(V_{33}, V_{11})$.

Remark 6.4. *As a corollary, we obtain the first term of the asymptotic expansion of* $\xi(\lambda, ; H_+, H_0)$ *near* $\lambda = \pm m$ *when*

- $W_\pm$ *admits a power-like decay at infinity,*
- $W_\pm$ *admits an exponential decay at infinity,*
- $W_\pm$ *has compact support.*

Generalised version of Levinson's Theorem relating the eigenvalues asymptotics of H_+ near $\pm\mathsf{m}$ to the scattering phase shift for the pair $(\mathsf{H}_+, \mathsf{H}_0)$:

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Corollary 6.5. *When* $W_\pm$ *admits a power-like decay at infinity, we have*

$$\lim_{\varepsilon \searrow 0} \frac{\xi(-m-\varepsilon; H_+, H_0)}{\xi(-m+\varepsilon; H_+, H_0)} = \frac{1}{2\cos\big(\pi/(\nu-1)\big)},$$

and when $W_\pm$ *admits an exponential decay at infinity or has compact support (and* $\nu > 4$*), we have*

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Generalised version of Levinson's Theorem relating the eigenvalues asymptotics of H_+ near $\pm m$ to the scattering phase shift for the pair (H_+, H_0):

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(These values seem "independent" of the Hamiltonian...
...Bruneau/Raikov 2012?)

7 Some hints on the proofs

1. Define the weighted resolvent

$$T(z) := V^{1/2}(H_0 - z)^{-1}V^{1/2}, \quad z \in \mathbb{C} \setminus \sigma(H_0),$$

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$$A(z) := \operatorname{Re} T(z), \qquad B(z) := \operatorname{Im} T(z).$$

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2. For each $\lambda \in (-\sqrt{m^2+\zeta}, \sqrt{m^2+\zeta}) \setminus \{\pm m\}$, the limits

$$A(\lambda + i0) := \lim_{\varepsilon \searrow 0} A(\lambda + i\varepsilon)$$

and

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exist in suitable Schatten-von Neumann classes.

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$$T(z) := V^{1/2}(H_0 - z)^{-1}V^{1/2}, \quad z \in \mathbb{C} \setminus \sigma(H_0),$$

   and

$$A(z) := \operatorname{Re} T(z), \qquad B(z) := \operatorname{Im} T(z).$$

2. For each $\lambda \in (-\sqrt{m^2+\zeta}, \sqrt{m^2+\zeta}) \setminus \{\pm m\}$, the limits

$$A(\lambda + i0) := \lim_{\varepsilon \searrow 0} A(\lambda + i\varepsilon)$$

   and

$$B(\lambda + i0) := \lim_{\varepsilon \searrow 0} B(\lambda + i\varepsilon)$$

   exist in suitable Schatten-von Neumann classes.

   (limiting absorption principle)

3. Due to a general result of [Pushnitski01], we have

$$\xi(\lambda; H_+, H_0) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{dt}{1+t^2} \, n_-\big(1; A(\lambda + i0) + tB(\lambda + i0)\big).$$

for $\lambda \in (-\sqrt{m^2 + \zeta}, \sqrt{m^2 + \zeta}) \setminus \{\pm m\}$.

3. Due to a general result of [Pushnitski01], we have

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for $\lambda \in (-\sqrt{m^2+\zeta}, \sqrt{m^2+\zeta}) \setminus \{\pm m\}$.

4. Introduce the projection onto the eigenspaces of H_0 for $\lambda = \pm m$:

$$P := \begin{pmatrix} p \otimes 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & p \otimes 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

5. Decompose $\mathsf{T}(z)$ as

$$\mathsf{T}(z) = \underbrace{\mathsf{V}^{1/2}(\mathsf{H}_0 - z)^{-1}\mathsf{P}\mathsf{V}^{1/2}}_{\equiv \mathsf{T}_{\mathsf{div}}(z)} + \underbrace{\mathsf{V}^{1/2}(\mathsf{H}_0 - z)^{-1}(1-\mathsf{P})\mathsf{V}^{1/2}}_{\equiv \mathsf{T}_{\mathsf{bound}}(z)},$$

and use the corresponding decomposition

$$\mathsf{T}(\lambda) = \mathsf{T}_{\mathsf{div}}(\lambda) + \mathsf{T}_{\mathsf{bound}}(\lambda)$$

for $\lambda \in (-\sqrt{m^2+\zeta}, \sqrt{m^2+\zeta}) \setminus \{\pm m\}$.

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   for $\lambda \in (-\sqrt{m^2+\zeta}, \sqrt{m^2+\zeta}) \setminus \{\pm m\}$.

6. Use "integral kernel technics" to determine the asymptotic behaviour of $\xi(\lambda, ; \mathsf{H}_+, \mathsf{H}_0)$ as $\lambda \to \pm m$.

   The divergence of $\xi(\lambda, ; \mathsf{H}_+, \mathsf{H}_0)$ is due to the Birman-Schwinger type operator $\mathsf{T}_{\mathsf{div}}(\lambda)$.

8 Some references

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