Spectral and scattering properties at thresholds for the Laplacian in a half-space with a periodic boundary condition

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The model

Let $V \in \mathsf{L}^\infty(\mathbb{R}, \mathbb{R})$ be a 2π-periodic function, and let

$$h^V : \mathcal{H}^1(\mathbb{R} \times \mathbb{R}_+) \times \mathcal{H}^1(\mathbb{R} \times \mathbb{R}_+) \to \mathbb{C}$$

be the closed lower semibounded form given by

$$\begin{aligned} h^V(\varphi, \psi) := \int_{\mathbb{R}\times\mathbb{R}_+} \mathrm{d}x_1 \mathrm{d}x_2 \Big\{ &\overline{(\partial_1\varphi)(x_1, x_2)}\,(\partial_1\psi)(x_1, x_2) \\ &+ \overline{(\partial_2\varphi)(x_1, x_2)}\,(\partial_2\psi)(x_1, x_2) \Big\} \\ &+ \int_{\mathbb{R}} \mathrm{d}x_1\, V(x_1)\,\overline{\varphi(x_1, 0)}\,\psi(x_1, 0). \end{aligned}$$

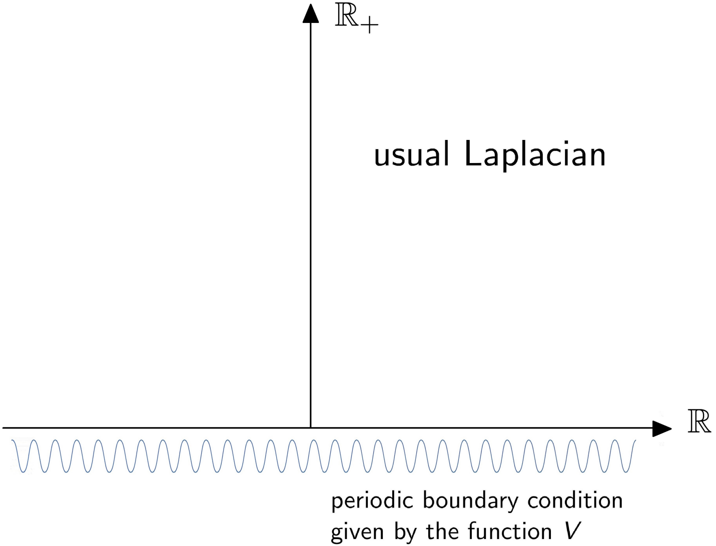
h^V induces a lower semibounded self-adjoint operator H^V with domain

$$\mathcal{D}(H^V) \subset \mathcal{H}^1(\mathbb{R} \times \mathbb{R}_+)$$

given by the relation

$$\left\langle H^V \varphi, \psi \right\rangle_{\mathsf{L}^2(\mathbb{R}\times\mathbb{R}_+)} = h^V(\varphi, \psi), \quad \varphi \in \mathcal{D}(H^V),\ \psi \in \mathcal{H}^1(\mathbb{R} \times \mathbb{R}_+).$$

If $V \equiv 0$, then H^0 is the Neumann Laplacian on $\mathbb{R} \times \mathbb{R}_+$.


$\mathbb{R}_+$
usual Laplacian
$\mathbb{R}$
periodic boundary condition
given by the function V

Due to the 2π-periodicity in the $\mathbb{R}$-variable, one uses a Bloch-Floquet-Gelfand transform

$$\mathcal{G} : \mathsf{L}^2(\mathbb{R} \times \mathbb{R}_+) \to \int_{[-1/2,1/2]}^{\oplus} \mathrm{d}k \ \mathsf{L}^2\left(\Pi\right), \quad \Pi := \mathbb{T} \times \mathbb{R}_+, \ \mathbb{T} := (-\pi, \pi),$$

to decompose H^V along the $\mathbb{R}$-variable.

One has

$$\mathcal{G}\, H^V \mathcal{G}^{-1} = \int_{[-1/2,1/2]}^{\oplus} \mathrm{d}k\, H_k^V ,$$

with H_k^V the self-adjoint operator associated to the form

$$h_k^V : \widetilde{\mathcal{H}}^1(\Pi)\times\widetilde{\mathcal{H}}^1(\Pi) \to \mathbb{C}, \quad \widetilde{\mathcal{H}}^1(\Pi) := \left\{ \begin{array}{c} \mathcal{H}^1\text{-functions with } 2\pi\text{-periodic} \\ \text{extension in the } \mathbb{R}\text{-variable} \end{array} \right\},$$

given by

$$\begin{aligned} h_k^V(\varphi,\psi) = \int_{\Pi} \mathrm{d}\theta\, \mathrm{d}x_2 \Big\{ & \overline{\big((-i\partial_1 + k)\varphi\big)(\theta,x_2)}\, \big((-i\partial_1 + k)\psi\big)(\theta,x_2) \\ & + \overline{(\partial_2\varphi)(\theta,x_2)}\, (\partial_2\psi)(\theta,x_2) \Big\} \\ & + \int_{\mathbb{T}} \mathrm{d}\theta\, V(\theta)\, \overline{\varphi(\theta,0)}\, \psi(\theta,0). \end{aligned}$$

If $V \equiv 0$, then H_k^0 reduces to

$$H_k^0 = (P+k)^2 \otimes 1 + 1 \otimes (-\triangle_{\mathrm{N}}),$$

where $P = -i\partial_1$ with periodic boundary condition on $\mathbb{T}$ and $-\triangle_{\mathrm{N}}$ the Neumann Laplacian on $\mathbb{R}_+$.

H_k^0 has purely a.c. spectrum $\sigma(H_k^0) = [k^2, \infty)$ and its spectral multiplicity is piecewise constant with a jump at each point of the threshold set

$$\tau_k := \{\lambda_{k,n}\}_{n \in \mathbb{Z}},$$

with $\lambda_{k,n} := (n+k)^2$ the n-th eigenvalue of $(P+k)^2$.

We write $\mathcal{P}_n$ for the 1-dimensional orthogonal projection in $\mathrm{L}^2(\mathbb{T})$ associated to $\lambda_{k,n}$.

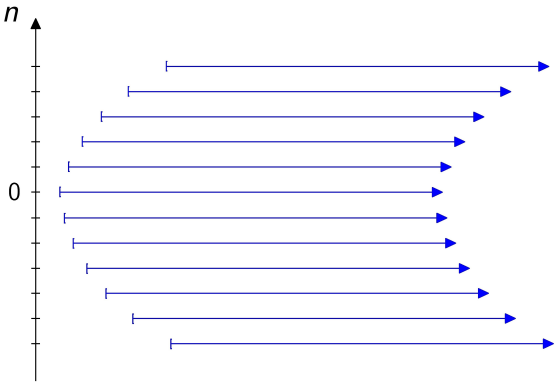
Basic spectral and scattering properties

Known facts ([Frank 2003] and [Frank/Shterenberg 2004]):

- The wave operators $W_{k,\pm} := \text{s-lim}_{t\to\pm\infty} \mathrm{e}^{itH_k^V}\,\mathrm{e}^{-itH_k^0}$ exist and are complete.
- The scattering operator $S_k := W_{k,+}^* W_{k,-}$ is unitary.
- $\sigma_{\mathrm{ac}}(H_k^V) = \sigma_{\mathrm{ac}}(H_k^0) = [k^2, \infty)$
- $\sigma_{\mathrm{sc}}(H_k^V) = \varnothing$
- The eigenvalues of H_k^V (if any) have finite multiplicity and may accumulate at $+\infty$ only.

S_k is decomposable in the spectral representation of H_k^0 as follows. Set

$$\mathscr{H}_k := \bigoplus_{n\in\mathbb{Z}} \mathscr{H}_{k,n} \quad \text{with} \quad \mathscr{H}_{k,n} := \mathsf{L}^2\big([\lambda_{k,n},\infty);\mathcal{P}_n\,\mathsf{L}^2(\mathbb{T})\big).$$



There is a unitary operator $\mathscr{U}_k : \mathsf{L}^2(\Pi) \to \mathscr{H}_k$ such that

$$\mathscr{U}_k H_k^0 \mathscr{U}_k^* = L_k \quad (L_k, \text{ multiplication operator in } \mathcal{H}_k)$$

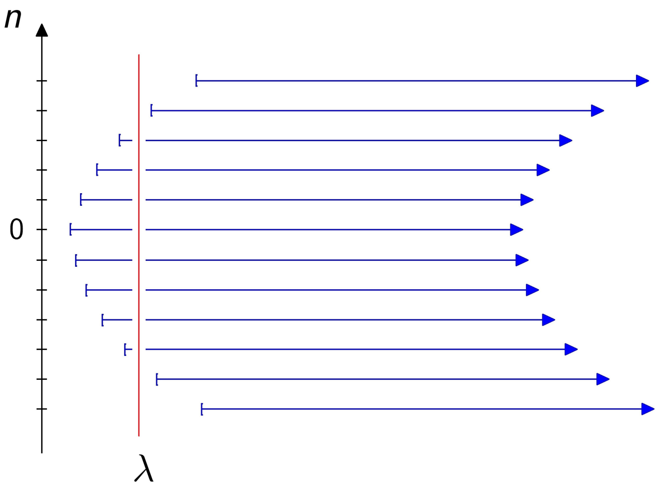
and

$$\left(\mathscr{U}_k S_k \mathscr{U}_k^* \xi\right)_n(\lambda) := \sum_{\{n' : \lambda_{k,n'} \leq \lambda\}} S_k(\lambda)_{nn'} \, \xi_{n'}(\lambda),$$

with $\lambda \in [\lambda_{k,n}, \infty) \setminus \{\tau_k \cup \sigma_{\mathrm{p}}(H_k^V)\}$ and

$$S_k(\lambda)_{nn'} \in \mathscr{B}_{n'n} := \mathscr{B}\big(\mathcal{P}_{n'} \, \mathsf{L}^2(\mathbb{T}), \mathcal{P}_n \, \mathsf{L}^2(\mathbb{T})\big)$$

the $n' \mapsto n$ channel scattering matrix.



n
0
λ

It is "easy" to show that the map

$$[k^2, \infty) \setminus \{\tau_k \cup \sigma_{\mathrm{p}}(H_k^V)\} \ni \lambda \mapsto S_k(\lambda)_{nn'} \in \mathscr{B}_{n'n}, \quad \lambda_{k,n}, \lambda_{k,n'} < \lambda.$$

is continuous.

So, it remains to determine the behaviour of $S_k(\lambda)_{nn'}$ as $\lambda \to \lambda_0 \in \tau_k \cup \sigma_{\mathrm{p}}(H_k^V)$.

Continuity of the scattering matrix

The behaviour of $S_k(\lambda)_{nn'}$ as $\lambda \to \lambda_0 \in \tau_k$ is as follows:

Theorem (Richard, T. 2017)

Let $\lambda_{k,m} \in \tau_k$ and $n, n' \in \mathbb{Z}$. Then,

(a) *If $\lambda_{k,n}, \lambda_{k,n'} < \lambda_{k,m}$, the map $\lambda \mapsto S_k(\lambda)_{nn'}$ is continuous in a neighbourhood of $\lambda_{k,m}$,*

(b) *If $\lambda_{k,n}, \lambda_{k,n'} \leq \lambda_{k,m}$, the limit $\lim_{\varepsilon \searrow 0} S_k(\lambda + \varepsilon)_{nn'}$ exists.*

- One cannot ask for more continuity in (b), since a channel could open at the energy $\lambda_{k,m}$.
- The case $\lambda \to \lambda_0 \in \sigma_{\mathrm{p}}(H_k^V)$ is easier to treat.

Idea of the proof.

We use a stationary formula

$$S_k(\lambda)_{nn'} = \delta_{nn'} - |\lambda - \lambda_{k,n}|^{-1/4} \mathcal{P}_n |V|^{1/2} \mathrm{M}_k(\lambda + i0) |V|^{1/2} \mathcal{P}_{n'} |\lambda - \lambda_{k,n'}|^{-1/4}.$$

with $\mathrm{M}_k(\lambda + i0) \in \mathscr{B}\big(\mathrm{L}^2(\mathbb{T})\big)$ the boundary value of a suitable weighted resolvent for H_k^V. Then, we prove an asymptotic expansion for $\mathrm{M}_k(\lambda + \varepsilon)$ for small $\varepsilon \in \mathbb{C}$.

(it is new and a "bit" technical because the thresholds $\lambda_{k,m}$ are embbeded in the a.c. spectrum of H_k^V)

□

New formulas for the wave operators

Using the asymptotic expansions for $\mathsf{M}_k(\lambda + i0)$ and stationary formulas for $W_{k,\pm}$, we obtain new formulas for $W_{k,\pm}$ in terms of the generator A_+ of dilations in $\mathsf{L}^2(\mathbb{R}_+)$:

Theorem (Richard, T. 2017)

We have in $\mathsf{L}^2(\Pi)$

$$W_{k,-} - 1 = \big(1 \otimes R(A_+)\big)(S_k - 1) + Q_k,$$

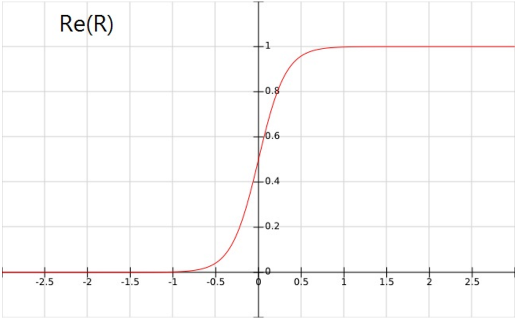
$$W_{k,+} - 1 = \big(1 - 1 \otimes R(A_+)\big)(S_k^* - 1) + Q_k S_k^*,$$

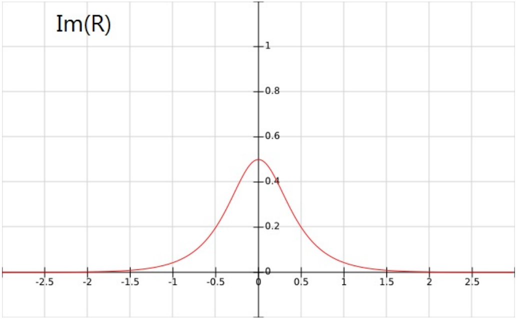
with

$$R(x) := \frac{1}{2}\big(1 + \tanh(\pi x) + i\cosh(\pi x)^{-1}\big), \quad x \in \mathbb{R}.$$

and

$$\operatorname*{s-lim}_{t \to \pm\infty} \mathrm{e}^{itH_k^0} Q_k \, \mathrm{e}^{-itH_k^0} = 0.$$


Re(R)
1
0.8
0.6
0.4
0.2
0
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5


Im(R)
1
0.8
0.6
0.4
0.2
0
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5

Two questions:

- Can we determine the behaviour of the operators $S_k(\lambda)_{nn'}$ as $\lambda \to \infty$? Do we have something like

$$\lim_{\lambda \to \infty} S_k(\lambda)_{nn'} = \delta_{nn'} \quad ?$$

- Can we prove that the operator Q_k is compact?

If that were the case, one could probably show the affiliation of the operators $W_{k,\pm} - 1$ to some appropriate C^*-algebra, and thus infer some index theorem relating $W_{k,\pm} - 1$ and S_k.

Using the direct integral decompositions

$$\mathcal{G}\, W_\pm\, \mathcal{G}^{-1} = \int_{[-1/2,1/2]}^{\oplus} \mathrm{d}k\, W_{k,\pm} \quad \text{and} \quad \mathcal{G}\, S\, \mathcal{G}^{-1} = \int_{[-1/2,1/2]}^{\oplus} \mathrm{d}k\, S_k,$$

we obtain an integrated version of the preceding formulas:

Theorem (Richard, T. 2017)

We have in $\mathrm{L}^2(\mathbb{R} \times \mathbb{R}_+)$

$$W_- - 1 = \big(1 \otimes R(A_+)\big)(S - 1) + Q,$$

$$W_+ - 1 = \big(1 - 1 \otimes R(A_+)\big)(S^* - 1) + Q\, S^*,$$

with $Q := \mathcal{G}^{-1} \left(\int_{[-1/2,1/2]}^{\oplus} \mathrm{d}k\, Q_k \right) \mathcal{G}$.

Thank you !

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