Commutator methods for the spectral analysis of time changes of horocycle flows

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Commutator methods

- $\mathcal{H}$, Hilbert space with norm $\|\cdot\|$ and scalar product $\langle\cdot,\cdot\rangle$
- $\mathscr{B}(\mathcal{H})$, set of bounded linear operators on $\mathcal{H}$
- $\mathscr{K}(\mathcal{H})$, set of compact operators on $\mathcal{H}$
- A, H, self-adjoint operators in $\mathcal{H}$ with domains $\mathcal{D}(A), \mathcal{D}(H)$, spectral measures $E^A(\cdot), E^H(\cdot)$ and spectra $\sigma(A), \sigma(H)$

Definition

$S \in \mathscr{B}(\mathcal{H})$ satisfies $S \in C^k(A)$ if

$$\mathbb{R} \ni t \mapsto \mathrm{e}^{-itA} S \, \mathrm{e}^{itA} \in \mathscr{B}(\mathcal{H})$$

is strongly of class C^k.

$S \in C^1(A)$ if and only if

$$\big|\langle A\varphi, S\varphi\rangle - \langle \varphi, SA\varphi\rangle\big| \le \mathrm{Const.} \, \|\varphi\|^2 \quad \text{for all } \varphi \in \mathcal{D}(A).$$

The operator corresponding to the continuous extension of the quadratic form is denoted by $[A, S] \in \mathscr{B}(\mathcal{H})$.

Definition

A self-adjoint operator H is of class $C^k(A)$ if $(H-z)^{-1} \in C^k(A)$ for some $z \in \mathbb{C} \setminus \sigma(H)$.

If H is of class $C^1(A)$, then

$$\big[A, (H-z)^{-1}\big] = (H-z)^{-1}\,[H, A]\,(H-z)^{-1},$$

with

$$[H, A] \in \mathscr{B}\big(\mathcal{D}(H), \mathcal{D}(H)^*\big)$$

the operator corresponding to the continuous extension to $\mathcal{D}(H)$ of the quadratic form

$$\mathcal{D}(H) \cap \mathcal{D}(A) \ni \varphi \mapsto \langle H\varphi, A\varphi\rangle - \langle A\varphi, H\varphi\rangle \in \mathbb{C}.$$

Theorem ([Mourre 81])

Let H be of class $C^2(A)$. Assume there exist an open set $I \subset \mathbb{R}$, a number $a > 0$ and $K \in \mathscr{K}(\mathcal{H})$ such that

$$E^H(I)[iH, A]E^H(I) \geq a\,E^H(I) + K. \qquad (\bigstar)$$

Then, H has at most finitely many eigenvalues in I (multiplicities counted), and H has no singular continuous spectrum in I.

- The inequality $(\bigstar)$ is called a Mourre estimate for H on I.
- The operator A is called a conjugate operator for H on I.
- If $K = 0$, then H is purely absolutely continuous in $I \cap \sigma(H)$.

Let M be a manifold with a probability measure μ, and let $\{F_t\}_{t\in\mathbb{R}}$ be a C^∞ measure preserving flow on M with complete vector field X.

Then, ergodicity, weak mixing and strong mixing of $\{F_t\}_{t\in\mathbb{R}}$ are expressible in terms of the self-adjoint operator $H := -iX$ in $\mathrm{L}^2(M,\mu)$:

- $\{F_t\}_{t\in\mathbb{R}}$ is ergodic iff 0 is a simple eigenvalue of H,
- $\{F_t\}_{t\in\mathbb{R}}$ is weakly mixing iff H has purely continuous spectrum in $\{\mathbb{C}\cdot 1\}^\perp$.
- $\{F_t\}_{t\in\mathbb{R}}$ is strongly mixing iff

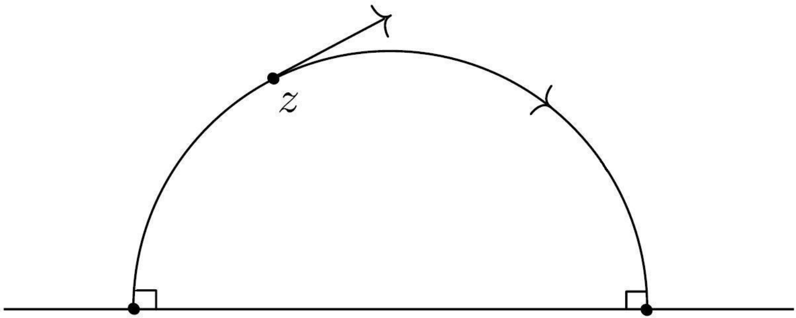
$$\lim_{t\to\infty} \langle \varphi, \mathrm{e}^{-itH)}\, \varphi \rangle = 0 \quad \text{for all } \varphi \in \{\mathbb{C}\cdot 1\}^\perp.$$

a.c. spectrum in $\{\mathbb{C}\cdot 1\}^\perp$	$\Rightarrow$	strong mixing	$\Rightarrow$	weak mixing	$\Rightarrow$	ergodicity

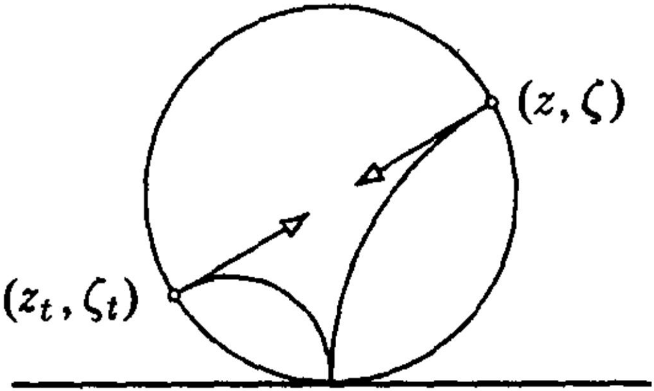
Horocycle flows

- Σ, compact Riemannian surface of constant negative curvature
- $M := T^1\Sigma$, unit tangent bundle of Σ
  (M is a compact 3-manifold with probability measure μ,
  $M \simeq \Gamma \setminus \mathrm{PSL}(2;\mathbb{R})$ for some cocompact lattice Γ in $\mathrm{PSL}(2;\mathbb{R})$)
- $F_1 \equiv \{F_{1,t}\}_{t\in\mathbb{R}}$, horocycle flow on M
- $F_2 \equiv \{F_{2,t}\}_{t\in\mathbb{R}}$, geodesic flow on M

The flows F_1 and F_2 are 1-parameter groups of diffeomorphisms preserving the measure μ.



Geodesic in the Poincaré half plane



(Positive) horocycle flow in the Poincaré half plane

The unitary group

$$U_j(t)\varphi := \varphi \circ F_{j,t}, \quad t \in \mathbb{R},\ \varphi \in \mathcal{H} := \mathsf{L}^2(M, \mu_\Omega),$$

has essentially self-adjoint generator

$$H_j\,\varphi := -i\,X_j\,\varphi, \quad \varphi \in C^\infty(M),$$

where X_j is the divergence-free vector field associated to F_j.

The horocycle flow F_1 is uniquely ergodic with respect to μ [Furstenberg 73], mixing of all orders [Marcus 78], and $U_1(t)$ has countable Lebesgue spectrum for each $t \neq 0$ [Parasyuk 53].

The horocycle flow and the geodesic flow satisfy the homogeneous commutation relation (see for instance [Bachir/Mayer 00])

$$U_2(s)\,U_1(t)\,U_2(-s) = U_1(\mathrm{e}^s\,t), \quad s,t \in \mathbb{R}, \qquad (\bigstar\bigstar)$$

which is a consequence of the matrix identity in $\mathrm{SL}(2,\mathbb{R})$:

$$\begin{pmatrix} \mathrm{e}^{s/2} & 0 \\ 0 & \mathrm{e}^{-s/2} \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-s/2} & 0 \\ 0 & \mathrm{e}^{s/2} \end{pmatrix} = \begin{pmatrix} 1 & \mathrm{e}^s\,t \\ 0 & 1 \end{pmatrix}.$$

Applying the strong derivatives $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}$ and $\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}$ in $(\bigstar\bigstar)$, one obtains that H_1 is of class $C^\infty(H_2)$ with

$$\big[iH_1, H_2\big] = H_1.$$

Time changes of horocycle flows

Take a C^1 vector field proportional to X_1; that is, $f X_1$ with $f \in C^1\big(M;(0,\infty)\big)$, and let $\widetilde{F}_1$ be the flow of $f X_1$.

The unitary group

$$\widetilde{U}_1(t)\varphi := \varphi \circ \widetilde{F}_{1,t}\,, \quad t \in \mathbb{R},\ \varphi \in \widetilde{\mathcal{H}} := \mathsf{L}^2(M, \mu_\Omega / f),$$

has generator $\widetilde{H} := -i f X_1$ essentially self-adjoint on $C^1(M)$ and unitarily equivalent to the operator in $\mathcal{H}$ given by

$$H := f^{1/2} H_1 f^{1/2}.$$

(The unitary $\mathscr{U} : \mathcal{H} \to \widetilde{\mathcal{H}}$, $\varphi \mapsto f^{1/2}\varphi$ realises the equivalence.)

What are the spectral properties of $\widetilde{H}$ (or equivalently of H)?

- Spectral properties are in general not preserved under time changes even though basic ergodic properties are preserved.
- In 1974, Kushnirenko shows that the flow $\widetilde{F}_1$ is strongly mixing if f is of class C^∞ and $f - X_2(f) > 0$. So, $\widetilde{H}$ has purely continuous spectrum in $\mathbb{R} \setminus \{0\}$ in this case.
- In 2006, Katok and Thouvenot conjecture that $\widetilde{H}$ has absolute continuous spectrum (even countable Lebesgue spectrum) if f is sufficiently smooth.

Mourre estimate

Let $z \in \mathbb{C} \setminus \mathbb{R}$ and assume for a moment that $f \equiv 1$, so that $H \equiv H_1$. Then, one has $(H+z)^{-1} \in C^1(H_2)$ with

$$\begin{aligned}\left[i(H+z)^{-1}, H_2\right] &= -(H+z)^{-1}[iH, H_2](H+z)^{-1} \\ &= -(H+z)^{-1}H(H+z)^{-1}.\end{aligned}$$

It follows that

$$\left[i\left(H^2+1\right)^{-1}, H_2\right] = -\left(H^2+1\right)^{-1} 2H^2 \left(H^2+1\right)^{-1}.$$

Thus H^2 is of class $C^\infty(H_2)$ with $[iH^2, H_2] = 2H^2$, and

$$E^{H^2}(I)\big[iH^2, H_2\big]E^{H^2}(I) = E^{H^2}(I)2H^2E^{H^2}(I) \geq 2\inf(I)\,E^{H^2}(I)$$

for each open bounded set $I \subset (0, \infty)$.

Therefore, in the case $f \equiv 1$, Mourre's theorem applies to the operator H^2 on the interval $(0, \infty)$.

So, let's try the same approach in the case $f \not\equiv 1\dots$

If $f \not\equiv 1$, one has $(H+z)^{-1} \in C^1(H_2)$ with

$$\left[i(H+z)^{-1}, H_2\right] = -(H+z)^{-1}(Hg+gH)(H+z)^{-1}$$

and

$$g := \frac{1}{2} - \frac{1}{2} X_2\big(\ln(f)\big).$$

$$\left(\text{note that } g = \frac{f - X_2(f)}{2f} > 0 \text{ under Kushnirenko's condition}\right)$$

This implies that $(H^2+1)^{-1} \in C^2(H_2)$ with

$$\left[iH^2, H_2\right] = H^2 g + 2HgH + gH^2.$$

Now, if $g > 0$ and f is of class C^2, one obtains that

$$H^2 g + gH^2 = HHg^{1/2}g^{1/2} + g^{1/2}g^{1/2}HH = \cdots = 2\big[H, g^{1/2}\big]^2 \geq 0,$$

and thus

$$\begin{aligned} E^{H^2}(I)\big[iH^2, H_2\big]E^{H^2}(I) &= E^{H^2}(I)\big(H^2 g + 2HgH + gH^2\big)E^{H^2}(I) \\ &\geq aE^{H^2}(I) \quad \text{with} \quad a := 2\inf(I) \cdot \inf_{p \in M} g(p) > 0 \end{aligned}$$

for each bounded open set $I \subset (0, \infty)$.

Since $(H^2 + 1)^{-1} \in C^2(H_2)$, we conclude by Mourre's theorem that H^2 is purely absolutely continuous outside $\{0\}$, where it has a simple eigenvalue corresponding to the constant functions.

$$\implies H \text{ has the same spectral properties as } H^2.$$

Summing up:

Theorem ([T. 2012])

Under Kushnirenko's condition, for time changes f of class C^2, the operator associated to the vector field fX_1 has purely absolutely continuous spectrum, except at 0, where it has a simple eigenvalue.

In fact, we also show this for noncompact surfaces Σ of finite volume.

Fine, but... [Forni/Ulcigrai 12] have obtained the same result (and also Lebesgue maximal spectral type) without assuming Kushnirenko's condition (for compact surfaces and for time changes in a Sobolev space of order $> 11/2$).

So, can we get rid of Kushnirenko's condition?

Mourre estimate (one more time)

Lemma (Conjugate operator)

Take $f \in C^3\big(M;(0,\infty)\big)$ *and* $L > 0$. *Then, the operator*

$$A_L\varphi := \frac{1}{L}\int_0^L \mathrm{d}t\ \mathrm{e}^{itH}\,H_2\,\mathrm{e}^{-itH}\,\varphi, \quad \varphi \in C^1(M),$$

is essentially self-adjoint in $\mathcal{H}$.

Idea of the proof. A calculation on $C^1(M)$ shows that

$$\frac{1}{L}\int_0^L \mathrm{d}t\ \mathrm{e}^{itH}\,H_2\,\mathrm{e}^{-itH} = -i\big(X + \tfrac{1}{2}\,\mathrm{div}_\Omega\,X\big),$$

for some vector field X on M. Furthermore, if f is of class C^3, then the r.h.s. is the self-adjoint generator of a strongly continuous unitary group (see [Abraham/Marsden 78]).

Replacing H_2 by A_L in the previous calculations and noting that

$$\begin{aligned} g_L := \frac{1}{L}\int_0^L \mathrm{d}t\ \mathrm{e}^{itH}\, g\, \mathrm{e}^{-itH} &= \frac{1}{L}\int_0^L \mathrm{d}t\ \mathrm{e}^{it\,\mathscr{U}^*\widetilde{H}\mathscr{U}}\ g\, \mathrm{e}^{-it\,\mathscr{U}^*\widetilde{H}\mathscr{U}} \\ &= \frac{1}{L}\int_0^L \mathrm{d}t\ \mathscr{U}^*\,\mathrm{e}^{it\widetilde{H}}\, g\, \mathrm{e}^{-it\widetilde{H}}\,\mathscr{U} \\ &= \frac{1}{L}\int_0^L \mathrm{d}t\,\big(g\circ\widetilde{F}_{1,-t}\big), \end{aligned}$$

we obtain that $(H^2+1)^{-1}\in C^2(A_L)$ with

$$\big[i(H^2+1)^{-1},A_L\big] = -(H^2+1)^{-1}\big(H^2 g_L + 2Hg_LH + g_LH^2\big)(H^2+1)^{-1}.$$

$\widetilde{F}_1$ is uniquely ergodic, since it is a reparametrisation of the uniquely ergodic flow $\{F_{1,t}\}_{t\in\mathbb{R}}$ [Humphries 74].

So, the Birkhoff average $g_L = \frac{1}{L}\int_0^L \mathrm{d}t\,\big(g \circ \widetilde{F}_{1,-t}\big)$ converges uniformly on M to $\int_M \mathrm{d}\widetilde{\mu}_\Omega\, g_L$; that is,

$$\begin{aligned}\lim_{L\to\infty} g_L = \int_M \mathrm{d}\widetilde{\mu}_\Omega\, g_L &= \frac{1}{2} - \frac{1}{2}\int_M \mathrm{d}\widetilde{\mu}_\Omega\, X_2\big(\ln(f)\big) \\ &= \frac{1}{2} + \frac{1}{2\int_M f^{-1}\,\mathrm{d}\mu_\Omega}\int_M \mathrm{d}\mu_\Omega\, X_2\big(f^{-1}\big) \\ &= \frac{1}{2} + \frac{i}{2\int_M f^{-1}\,\mathrm{d}\mu_\Omega}\big\langle 1, H_2 f^{-1}\big\rangle \\ &= \frac{1}{2}.\end{aligned}$$

$$\implies g_L > 0 \text{ if } L > 0 \text{ is large enough.}$$

So, we got rid off Kushnirenko's condition, and have proved the following:

Theorem ([T. 2014])

For time changes f of class C^3, the operator associated to the vector field fX_1 has purely absolutely continuous spectrum, except at 0, where it has a simple eigenvalue.

(. . . if someone knows how to prove Lebesgue spectrum . . .)

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