Spectral properties of horocycle flows for compact surfaces of constant negative curvature

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Commutator methods

- $\mathcal{H}$, Hilbert space with norm $\|\cdot\|$ and scalar product $\langle\cdot,\cdot\rangle$
- $\mathscr{B}(\mathcal{H})$, set of bounded linear operators on $\mathcal{H}$
- $\mathscr{K}(\mathcal{H})$, set of compact operators on $\mathcal{H}$
- A, H, self-adjoint operators in $\mathcal{H}$ with domains $\mathcal{D}(A), \mathcal{D}(H)$, spectral measures $E^A(\cdot), E^H(\cdot)$, and spectra $\sigma(A), \sigma(H)$

Definition

$S \in \mathscr{B}(\mathcal{H})$ satisfies $S \in C^k(A)$ if the map

$$\mathbb{R} \ni t \mapsto \mathrm{e}^{-itA} \, S \, \mathrm{e}^{itA} \in \mathscr{B}(\mathcal{H})$$

is strongly of class C^k.

$S \in C^1(A)$ if and only if

$$\big| \langle \varphi, S\,A\varphi \rangle - \langle A\varphi, S\varphi \rangle \big| \leq \text{Const.} \, \|\varphi\|^2 \quad \text{for all } \varphi \in \mathcal{D}(A).$$

The bounded operator associated to the continuous extension of the quadratic form is denoted by $[S, A]$, and

$$[iS, A] = \text{s-} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{e}^{-itA} \, S \, \mathrm{e}^{itA} \in \mathscr{B}(\mathcal{H}).$$

Definition

A self-adjoint operator H is of class $C^k(A)$ if $(H-z)^{-1} \in C^k(A)$ for some $z \in \mathbb{C} \setminus \sigma(H)$.

If H is of class $C^1(A)$, then

$$\left[A, (H-z)^{-1}\right] = (H-z)^{-1}[H, A](H-z)^{-1},$$

with $[H, A]$ the operator from $\mathcal{D}(H)$ to $\mathcal{D}(H)^*$ corresponding to the continuous extension to $\mathcal{D}(H)$ of the quadratic form

$$\mathcal{D}(H) \cap \mathcal{D}(A) \ni \varphi \mapsto \langle H\varphi, A\varphi \rangle - \langle A\varphi, H\varphi \rangle \in \mathbb{C}.$$

Theorem (Mourre 1981, and others in the 1990's)

Let H be of class $C^2(A)$. Assume there exist a bounded Borel set $I \subset \mathbb{R}$, a number $a > 0$ and $K \in \mathscr{K}(\mathcal{H})$ such that

$$E^H(I)[iH, A]E^H(I) \geq a E^H(I) + K. \qquad (\bigstar)$$

Then, H has at most finitely many eigenvalues in I (multiplicities counted), and H has no singular continuous spectrum in I.

- The inequality $(\bigstar)$ is called a Mourre estimate for H on I.
- The operator A is called a conjugate operator for H on I.
- If $K = 0$, H has purely absolutely continuous spectrum in $I \cap \sigma(H)$.

Flows

Let M be a smooth manifold with probability measure μ, and $\{F_t\}_{t\in\mathbb{R}}$ a C^1 measure preserving flow on M with C^0 vector field X_F.

Ergodicity, weak mixing and strong mixing of $\{F_t\}_{t\in\mathbb{R}}$ with respect to μ are expressible in terms of the self-adjoint operator $H_F := iX_F$ in $\mathrm{L}^2(M,\mu)$.

- $\{F_t\}_{t\in\mathbb{R}}$ is ergodic if and only if 0 is a simple eigenvalue of H,
- $\{F_t\}_{t\in\mathbb{R}}$ is weakly mixing if and only if H_F has purely continuous spectrum in $\mathbb{R}\setminus\{0\}$.
- $\{F_t\}_{t\in\mathbb{R}}$ is strongly mixing if and only if

$$\lim_{t\to\infty}\left\langle\varphi,\mathrm{e}^{-itH_F}\varphi\right\rangle=0 \quad \text{for all } \varphi\in\{\mathbb{C}\cdot 1\}^{\perp}.$$

a.c. spectrum in $\{\mathbb{C}\cdot 1\}^{\perp}$ $\Rightarrow$ strong mixing $\Rightarrow$ weak mixing $\Rightarrow$ ergodicity

Minimal W^{u} flows

- M, compact connected Riemannian manifold with distance d,
- $\{f_t\}_{t\in\mathbb{R}}$, $C^{1+\varepsilon}$ Anosov flow on M; that is, a $C^{1+\varepsilon}$ flow on M without fixed points, with three submanifolds $W^{\mathrm{u}}(x)$, $W^{\mathrm{s}}(x)$, $\mathrm{Orb}(x)$ passing through each $x \in M$,

$$W^{\mathrm{u}}(x) = \left\{ y \in M \mid \lim_{t\to-\infty} d\big(f_t(x), f_t(y)\big) = 0 \right\} \quad \text{(unstable manifold)},$$
$$W^{\mathrm{s}}(x) = \left\{ y \in M \mid \lim_{t\to+\infty} d\big(f_t(x), f_t(y)\big) = 0 \right\} \quad \text{(stable manifold)},$$
$$\mathrm{Orb}(x) = \left\{ f_t(x) \mid t \in \mathbb{R} \right\} \quad \text{(orbit)},$$

with respective tangent spaces E_x^{u}, E_x^{s}, E_x continuous in x and satisfying

$$T_x M = E_x^{\mathrm{u}} \oplus E_x^{\mathrm{s}} \oplus E_x.$$

The flow $\{f_t\}_{t\in\mathbb{R}}$ has a C^{ε} vector field X_f.

Assume that $\{f_t\}_{t\in\mathbb{R}}$ is a codimension 1 Anosov flow. More specifically:

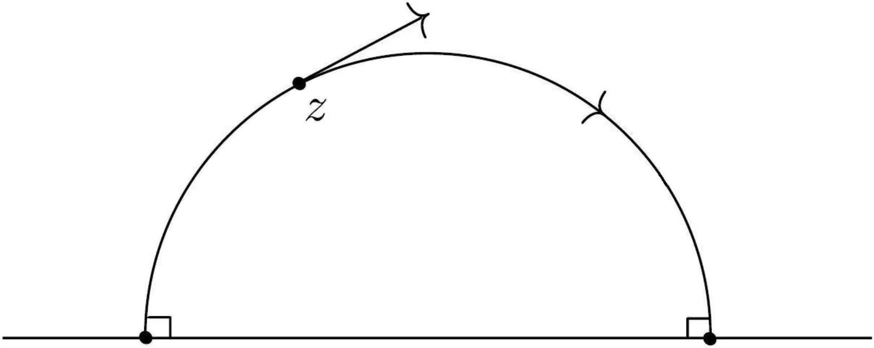
$\{W^{\mathrm{u}}(x)\}_{x\in M}$ is a 1-dimensional orientable C^0 foliation of M (in particular each $W^{\mathrm{u}}(x)$ is a curve), which supports a C^0 minimal flow $\{\phi_s\}_{s\in\mathbb{R}}$ whose orbits are the unstable manifolds.[1]

$\{\phi_s\}_{s\in\mathbb{R}}$ is called minimal W^{u} flow or minimal W^{u} parametrisation.

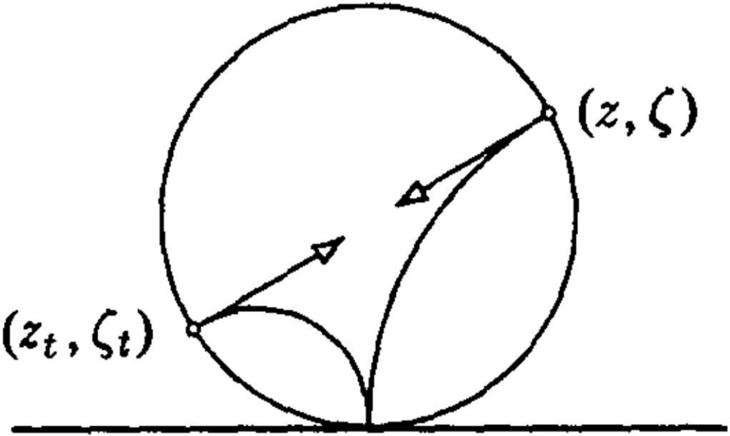
[1] A flow on a compact metric manifold is minimal if each of its orbit is dense.

Example

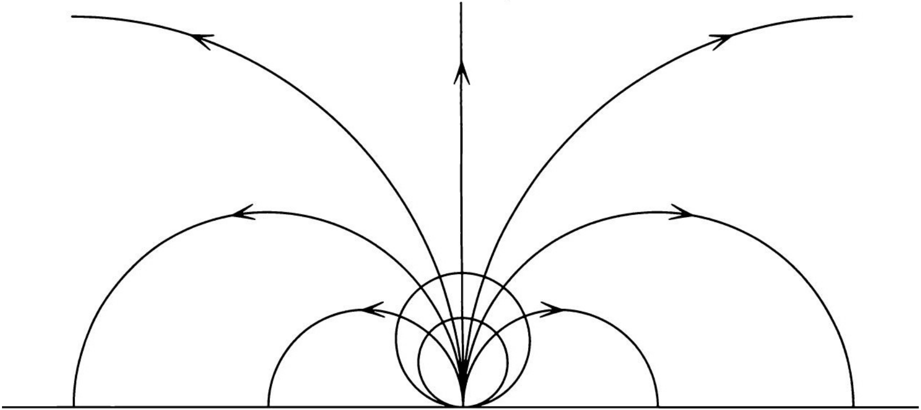
The iconic example of Anosov flow $\{f_t\}_{t\in\mathbb{R}}$ *and* W^{u} *flow* $\{\phi_s\}_{s\in\mathbb{R}}$ *are the geodesic flow and the horocycle flow on the unit tangent bundle of a compact connected orientable surface of (possibly variable) negative curvature.*



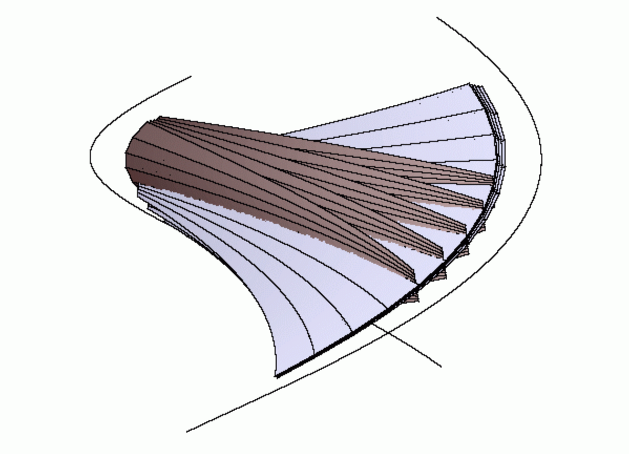
Geodesic flow in the Poincaré half plane



Positive horocycle flow in the Poincaré half plane
(from Bekka/Mayer's book)



Geodesics and horocycles in the Poincaré half plane
(from Hasselblatt/Katok's book)



Anosov stable and unstable foliations for the geodesic flow on the unit tangent bundle of a surface of constant negative curvature

(from http://kyokan.ms.u-tokyo.ac.jp/~showroom/)

Some facts from [Marcus 75], [Marcus 77], [Bowen-Marcus 77]:

(i) $\{\phi_s\}_{s\in\mathbb{R}}$ is uniquely ergodic w.r.t. a probability mesure μ on M.

This means that for any $h \in C(M)$ we have

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \mathrm{d}s \, \big(h \circ \phi_s\big)(x) = \int_M \mathrm{d}\mu(y)\, h(y)$$

uniformly in $x \in M$.

(ii) There exists $s^* : \mathbb{R} \times \mathbb{R} \times M \to \mathbb{R}$ such that

$$\big(f_t \circ \phi_s \circ f_{-t}\big)(x) = \phi_{s^*(t,s,x)}(x), \quad s,t \in \mathbb{R},\ x \in M,$$

(the Anosov flow $\{f_t\}_{t\in\mathbb{R}}$ expands the W^{u} orbits).

(iii) $\{W^u(x)\}_{x\in M}$ admits a C^0 parametrisation $\{\widetilde{\phi}_s\}_{s\in\mathbb{R}}$ such that

$$f_t \circ \widetilde{\phi}_s \circ f_{-t} = \widetilde{\phi}_{\lambda^t s}, \quad s,t \in \mathbb{R},\ \lambda > 1 \ \text{ (that is, } s^*(t,s,x) = \lambda^t s)$$

(uniformly expanding parametrisation).

(iv) $\{\widetilde{\phi}_s\}_{s\in\mathbb{R}}$ is uniquely ergodic w.r.t. a probability measure $\widetilde{\mu}$ given in terms of μ.

(v) $\widetilde{\mu}$ is invariant under the Anosov flow $\{f_t\}_{t\in\mathbb{R}}$.

Assumption 1

$\{\phi_s\}_{s\in\mathbb{R}}$ is C^1, and $\{\widetilde{\phi}_s\}_{s\in\mathbb{R}}$ is a C^1 reparametristation of $\{\phi_s\}_{s\in\mathbb{R}}$.

Under this assumption, we have:

- $\widetilde{\mu} = \mu/\widetilde{\rho}$ with $\widetilde{\rho} = \rho \int_M \mathrm{d}\mu\, \rho^{-1}$ and $\rho \in C\big(M; (0,\infty)\big)$.
- The group in $\mathcal{H} := \mathsf{L}^2(M,\mu)$ given by

  $$U_s^{\phi}\varphi := \varphi \circ \phi_s, \quad s \in \mathbb{R},\ \varphi \in \mathcal{H},$$

  is strongly continuous, unitary, with essentially self-adjoint generator

  $$H_\phi \varphi = iX_\phi \varphi, \quad \varphi \in C^1(M),$$

- The group in $\mathcal{H}$ given by

  $$U_t^{f}\varphi := \varphi \circ f_t, \quad t \in \mathbb{R},\ \varphi \in \mathcal{H},$$

  is strongly continuous, but not unitary if $\rho \not\equiv 1$.

Assumption 2

The derivative

$$u_{t,s}(x) := \big(\partial_1\partial_2 s^*\big)(t,s,x)$$

exists and is continuous in $s, t \in \mathbb{R}$ and $x \in M$.

Under this assumption, using the unique ergodicity of $\{\phi_s\}_{s\in\mathbb{R}}$, Marcus has proved that $\{\phi_s\}_{s\in\mathbb{R}}$ is strongly mixing w.r.t. μ. Therefore,

H_ϕ has purely continuous spectrum in $\mathbb{R} \setminus \{0\}$.

So, let's prove that H_ϕ has *purely absolutely continuous spectrum* in $\mathbb{R} \setminus \{0\}$ under some additional regularity assumption.

Mourre estimate

Assumption 3

X_f and X_ϕ are C^1, $X_f(\rho) \in C(M)$ and $\rho^{-1}X_f(\rho) \in C^1(M)$.

Intuitively, the conjugate operator is constructed as follows:

1) Sum $2iX_f$ and its "divergence" $i\rho^{-1}X_f(\rho)$ to get a symmetric operator $2iX_f + i\rho^{-1}X_f(\rho)$ on $C^1(M)$.
2) Take the Birkhoff average of $2iX_f + i\rho^{-1}X_f(\rho)$ along the flow $\{\phi_s\}_{s\in\mathbb{R}}$ to take into account the unique ergodicity of $\{\phi_s\}_{s\in\mathbb{R}}$.

Proposition (Conjugate operator)

Suppose that Assumptions 1, 2, 3 are satisfied. Then, the operator

$$A_t\,\varphi := \frac{1}{t}\int_0^t \mathrm{d}s\, U_s^{\phi}\big(2iX_f + i\rho^{-1}X_f(\rho)\big)\,U_{-s}^{\phi}\,\varphi, \quad t>0,\ \varphi\in C^1(M),$$

is essentially self-adjoint in $\mathcal{H}$.

Idea of the proof.

The operator $2iX_f + i\rho^{-1}X_f(\rho)$ is symmetric on $C^1(M)$, and the operations $U_s^{\phi}(\,\cdots)U_{-s}^{\phi}$ and $\frac{1}{t}\int_0^t \mathrm{d}s\,(\,\cdots)$ preserve this property. So, A_t is symmetric on $C^1(M)$.

Furthermore, A_t can be written as $i(X_t + g_t)$ on $C^1(M)$, with X_t a C^1 vector field and $g_t \in C^1(M;\mathbb{R})$.

Operators of this type are essentially self-adjoint on $C^1(M)$. □

With some calculations on $C^1(M)$ using properties of the flows $\{f_t\}_{t\in\mathbb{R}}$, $\{\phi_s\}_{s\in\mathbb{R}}$, $\{\widetilde{\phi}_s\}_{s\in\mathbb{R}}$ and the function $u_{t,s}(x) = \big(\partial_1\partial_2 s^*\big)(t,s,x)$, we obtain the following:

Lemma (Regularity of H_ϕ)

Suppose that Assumptions 1, 2, 3 are satisfied. Then, for $t > 0$ we have $(H_\phi - i)^{-1} \in C^2(A_t)$, and

$$\big[i\big(H_\phi - i\big)^{-1}, A_t\big] = 2\,\big(H_\phi - i\big)^{-1} c_t H_\phi \big(H_\phi - i\big)^{-1} - \big[\big(H_\phi - i\big)^{-1}, c_t\big]$$

with

$$c_t := \frac{1}{t}\int_0^t \mathrm{d}s\,\big(u_{0,0}\circ\phi_s\big).$$

Because of the general formula

$$\big[i(H-z)^{-1}, A\big] = -(H-z)^{-1}[iH, A](H-z)^{-1},$$

we infer from the lemma that

$$\begin{aligned}
&E^{H_\phi}(I)\big[iH_\phi, -A_t\big]E^{H_\phi}(I)\\
&= 2\,E^{H_\phi}(I)\,c_t H_\phi E^{H_\phi}(I) - \big(H_\phi - i\big)E^{H_\phi}(I)\big[\big(H_\phi - i\big)^{-1}, c_t\big]\big(H_\phi - i\big)E^{H_\phi}(I)
\end{aligned}$$

for each bounded Borel set $I \subset \mathbb{R}$.

Can we get some positivity out of the last equation ?

Proposition (Mourre estimate)

Suppose that Assumptions 1, 2, 3 are satisfied, and take $I \subset (0,\infty)$ compact with $I \cap \sigma(H_\phi) \neq \varnothing$. Then, there exist $t > 0$ and $a > 0$ such that

$$E^{H_\phi}(I)\big[iH_\phi, -A_t\big]E^{H_\phi}(I) \geq a\,E^{H_\phi}(I).$$

A similar result holds for $I \subset (-\infty, 0)$.

Idea of the proof.

The unique ergodicity of $\{\phi_s\}_{s\in\mathbb{R}}$ w.r.t. μ implies that

$$\lim_{t\to\infty} c_t = \lim_{t\to\infty} \frac{1}{t}\int_0^t \mathrm{d}s\,\big(u_{0,0}\circ\phi_s\big) = \int_M \mathrm{d}\mu\, u_{0,0}$$

uniformly on M. Moreover, some calculations show that

$$\int_M \mathrm{d}\mu\, u_{0,0} = \ln(\lambda) > 0.$$

Idea of the proof (continued).

So, one has for $t > 0$ large enough

$$
\begin{aligned}
&E^{H_\phi}(I)\big[iH_\phi, -A_t\big]E^{H_\phi}(I) \\
&= 2\,E^{H_\phi}(I)\,c_t H_\phi E^{H_\phi}(I) \\
&\quad - \big(H_\phi - i\big)E^{H_\phi}(I)\Big[\big(H_\phi - i\big)^{-1}, c_t - \ln(\lambda)\Big]\big(H_\phi - i\big)E^{H_\phi}(I) \\
&\approx 2\,E^{H_\phi}(I)\ln(\lambda)H_\phi E^{H_\phi}(I) \\
&\geq 2\ln(\lambda)\inf(I)\,E^{H_\phi}(I)
\end{aligned}
$$

which gives

$$
E^{H_\phi}(I)\big[iH_\phi, -A_t\big]E^{H_\phi}(I) \geq a\,E^{H_\phi}(I) \quad \text{with} \quad a \in \big(0, 2\ln(\lambda)\inf(I)\big).
$$

□

Using Mourre's theorem, we conclude that:

Theorem (Absolutely continuous spectrum)

Suppose that Assumptions 1, 2, 3 are satisfied. Then, H_ϕ has purely absolutely continuous spectrum, except at 0, where it has a simple eigenvalue with eigenspace $\mathbb{C} \cdot 1$.

- The theorem applies in particular to generators of reparametrisations of the horocycle flow on the unit tangent bundle of a compact connected orientable surface of **constant** negative curvature.
- For reparametrisations of the horocycle flow on the unit tangent bundle of a compact connected orientable surface of **variable** negative curvature the question is open.

Gracias !

References

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