Two-Hilbert spaces Mourre theory for the Laplace-Beltrami operator on manifolds with asymptotically cylindrical ends

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Mourre theory in one Hilbert space

- $\mathcal{H}$, Hilbert space with norm $\|\cdot\|$ and scalar product $\langle\cdot,\cdot\rangle$
- $\mathscr{B}(\mathcal{H})$, set of bounded linear operators on $\mathcal{H}$
- $\mathscr{K}(\mathcal{H})$, set of compact operators on $\mathcal{H}$
- A, H, self-adjoint operators in $\mathcal{H}$ with domains $\mathcal{D}(A), \mathcal{D}(H)$, spectral measures $E^A(\cdot), E^H(\cdot)$ and spectra $\sigma(A), \sigma(H)$

Definition

$S \in \mathscr{B}(\mathcal{H})$ satisfies $S \in C^k(A)$ if the map

$$\mathbb{R} \ni t \mapsto \mathrm{e}^{-itA} S \, \mathrm{e}^{itA} \in \mathscr{B}(\mathcal{H})$$

is strongly of class C^k.

$S \in C^1(A)$ if and only if

$$\big|\langle \varphi, S A \varphi\rangle - \langle A\varphi, S\varphi\rangle\big| \leq \text{Const.} \, \|\varphi\|^2 \quad \text{for all } \varphi \in \mathcal{D}(A).$$

The bounded operator associated to the continuous extension of the quadratic form is denoted by $[S, A]$, and

$$[iS, A] = \text{s-} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{e}^{-itA} S \, \mathrm{e}^{itA} \in \mathscr{B}(\mathcal{H}).$$

Definition

A self-adjoint operator H is of class $C^k(A)$ if $(H-z)^{-1} \in C^k(A)$ for some $z \in \mathbb{C} \setminus \sigma(H)$.

If H is of class $C^1(A)$, then

$$\left[A, (H-z)^{-1}\right] = (H-z)^{-1}[H, A](H-z)^{-1},$$

with $[H, A]$ the operator from $\mathcal{D}(H)$ to $\mathcal{D}(H)^*$ corresponding to the continuous extension to $\mathcal{D}(H)$ of the quadratic form

$$\mathcal{D}(H) \cap \mathcal{D}(A) \ni \varphi \mapsto \langle H\varphi, A\varphi \rangle - \langle A\varphi, H\varphi \rangle \in \mathbb{C}.$$

Mourre's theorem

Theorem (Mourre 81)

Let H be of class $C^2(A)$. Assume there exist an open set $I \subset \mathbb{R}$, a number $a > 0$ and $K \in \mathscr{K}(\mathcal{H})$ such that

$$E^H(I)[iH, A]E^H(I) \geq a E^H(I) + K. \qquad (\bigstar)$$

Then, H has at most finitely many eigenvalues in I (multiplicities counted), and H has no singular continuous spectrum in I.

- The inequality $(\bigstar)$ is called a Mourre estimate for H on I.
- The operator A is called a conjugate operator for H on I.
- If $K = 0$, then H is purely absolutely continuous in $I \cap \sigma(H)$.

First example in one Hilbert space

Let Q and P be the position and momentum operators in $\mathcal{H} := \mathrm{L}^2(\mathbb{R})$

$$(Q\varphi)(x) := x\,\varphi(x), \quad (P\varphi)(x) := -i\,(\partial_x\varphi)(x), \quad \varphi \in \mathscr{S}(\mathbb{R}),\ x \in \mathbb{R}.$$

Let H be the Laplacian

$$H := P^2 = -\partial_x^2, \quad \mathcal{D}(H) = \mathcal{H}^2(\mathbb{R}).$$

Let $A := \frac{1}{2}(PQ + QP)$ be the generator of the dilations group

$$\big(U(t)\varphi\big)(x) := \mathrm{e}^{t/2}\,\varphi(\mathrm{e}^t\,x), \quad \varphi \in \mathscr{S}(\mathbb{R}),\ x, t \in \mathbb{R}.$$

(A is essentially self-adjoint on $\mathscr{S}(\mathbb{R})$)

On $\mathscr{S}(\mathbb{R})$:

$$[H, A] = \frac{1}{2}\left[P^2, PQ + QP\right] = \frac{1}{2}\left\{P\,\underbrace{[P^2, Q]}_{-2iP} + \underbrace{[P^2, Q]}_{-2iP}\,P\right\} = -2iH.$$

Thus,

$$\begin{aligned}\left[(H+i)^{-1}, A\right] &= -(H+i)^{-1}\,[H, A]\,(H+i)^{-1} \\ &= 2i\,(H+i)^{-1}\,H\,(H+i)^{-1} \\ &\in \mathscr{B}(\mathcal{H})\end{aligned}$$

and

$$\left[\left[(H+i)^{-1}, A\right], A\right] \in \mathscr{B}(\mathcal{H}).$$

(first assumption of Mourre satisfied)

Let $b > a > 0$. Then,

$$\begin{aligned} E^H\big((a,b)\big)\,[iH,A]\,E^H\big((a,b)\big) &= 2E^H\big((a,b)\big)\,H\,E^H\big((a,b)\big) \\ &= 2E^H\big((a,b)\big)\int_{\sigma(H)} \lambda\,\chi_{(a,b)}(\lambda)\,E^H(\mathrm{d}\lambda) \\ &> 2E^H\big((a,b)\big)\,a\,E^H\big((a,b)\big) \\ &= 2a\,E^H\big((a,b)\big). \end{aligned}$$

(second assumption of E. Mourre satisfied with $K = 0$ for each interval $(a,b) \subset (0,\infty)$)

$\leadsto$ H is purely absolutely continuous on $(0,\infty)$.

Second example in one Hilbert space

Δ_Σ, Laplace-Beltrami operator on a compact, orientable, connected, Riemannian manifold $\left(\Sigma, h\right)$ of dimension $n \geq 1$ without boundary, with metric h and volume element ds.

Δ_Σ has discrete spectrum

$$0 = \tau_0 < \tau_1 \leq \tau_2 \leq \dots$$

So,

$$\Delta_\Sigma = \sum_{j \geq 0} \tau_j \mathcal{P}_j,$$

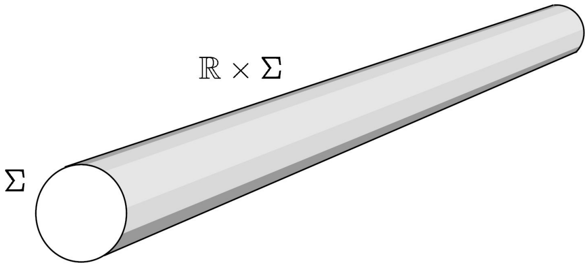
with $\mathcal{P}_j$ the 1D-orthogonal projection associated to τ_j.

(Hodge Theorem for functions)

Let

$$H_0 := \Delta_{\mathbb{R}\times\Sigma} \simeq P^2 \otimes 1 + 1 \otimes \Delta_\Sigma$$

be the Laplace-Beltrami operator in $\mathcal{H}_0 := \mathrm{L}^2(\mathbb{R}, \mathrm{d}x) \otimes \mathrm{L}^2(\Sigma, \mathrm{ds})$.



H_0 is essentially self-adjoint on $\mathscr{S}(\mathbb{R}) \odot C^\infty(\Sigma)$, and

$$A_0 := \frac{1}{2}(PQ + QP) \otimes 1$$

is essentially self-adjoint on $\mathscr{S}(\mathbb{R}) \odot \mathrm{L}^2(\Sigma, \mathrm{ds})$.

On $\mathscr{S}(\mathbb{R}) \odot C^\infty(\Sigma)$:

$$\begin{aligned}
[H_0, A_0] &= \left[P^2 \otimes 1 + 1 \otimes \Delta_\Sigma, \frac{1}{2}(PQ + QP) \otimes 1\right] \\
&= \left[P^2, \frac{1}{2}(PQ + QP)\right] \otimes 1 \\
&= -2iP^2 \otimes 1.
\end{aligned}$$

Thus,

$$\begin{aligned}
\left[(H_0 + i)^{-1}, A_0\right] &= -(H_0 + i)^{-1} [H_0, A_0] (H_0 + i)^{-1} \\
&= 2i (H_0 + i)^{-1} \big(P^2 \otimes 1\big)(H_0 + i)^{-1} \\
&\in \mathscr{B}(\mathcal{H}_0)
\end{aligned}$$

and

$$\left[\left[(H_0 + i)^{-1}, A_0\right], A_0\right] \in \mathscr{B}(\mathcal{H}_0).$$

(first assumption of Mourre satisfied)

Let $\tau_{j_0} < a < b < \tau_{j_0+1}$ for some $j_0 \in \mathbb{N}$. Then,

$$\begin{aligned}
&E^{H_0}\big((a,b)\big)\,[iH_0, A_0]\,E^{H_0}\big((a,b)\big)\\
&= E^{H_0}\big((a,b)\big)\big(2P^2 \otimes 1\big)E^{H_0}\big((a,b)\big)\\
&= 2E^{H_0}\big((a,b)\big)\,H_0\,E^{H_0}\big((a,b)\big)\\
&\quad - 2E^{H_0}\big((a,b)\big)\big(1 \otimes \Delta_\Sigma\big)E^{H_0}\big((a,b)\big)\\
&= 2\underbrace{E^{H_0}\big((a,b)\big)\,H_0\,E^{H_0}\big((a,b)\big)}_{> a\,E^{H_0}\left((a,b)\right)}\\
&\quad \underbrace{- 2\sum_{j\geq 0}\tau_j E^{H_0}\big((a,b)\big)\big(1 \otimes \mathcal{P}_j\big)E^{H_0}\big((a,b)\big)}_{\leq \tau_{j_0}\,E^{H_0}\left((a,b)\right)}\\
&> 2\,(a - \tau_{j_0})E^{H_0}\big((a,b)\big)
\end{aligned}$$

(second assumption of Mourre satisfied with $K = 0$ for each interval $(a, b) \subset (\tau_j, \tau_{j+1})$)

$\rightsquigarrow$ H_0 is purely absolutely continuous on $[0, \infty) \setminus \{\tau_j\}_{j \geq 0}$.

Mourre theory in a two-Hilbert spaces setting

- H, self-adjoint operator with domain $\mathcal{D}(H)$, spectral measure $E^H(\cdot)$, and spectrum $\sigma(H)$ in a first Hilbert space $\mathcal{H}$
- A_0, H_0, self-adjoint operators in a second Hilbert space $\mathcal{H}_0$
- $J \in \mathscr{B}(\mathcal{H}_0, \mathcal{H})$, identification operator

Theorem (Richard-T. 2013)

Assume that

(i) H_0 *is of class* $C^1(A_0)$ *in* $\mathcal{H}_0$,

(ii) *there is* $\mathscr{D} \subset \mathcal{D}(A_0 J^*) \subset \mathcal{H}$ *such that* JA_0J^* *is essentially self-adjoint on* $\mathscr{D}$, *with self-adjoint extension denoted by* A,

(iii) *compacity conditions between* H, H_0, A_0 *and* J.

Then, if H_0 *satisfies a Mourre estimate with respect to* A_0 *on* $I \subset \mathbb{R}$, H *also satisfies a Mourre estimate with respect to* A *on* $I \subset \mathbb{R}$.

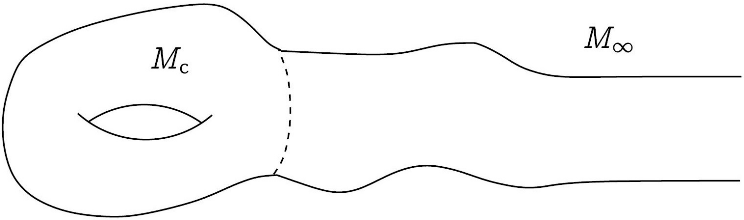
Thus, H has at most finitely many eigenvalues in I (multiplicities counted), and H has no singular continuous spectrum in I if H_0 and A_0 satisfy the two assumptions of Mourre on I.

Manifolds with asymptotically cylindrical ends

(M, g) complete Riemannian manifold of dimension $n+1 \geq 2$ without boundary, with metric g and volume element dv.

$M = M_{\rm c} \cup M_\infty$, with $M_{\rm c}$ relatively compact and M_∞ open in M with a diffeomorphism

$$\iota : M_\infty \to (0, \infty) \times \Sigma.$$



- $H := \Delta_M$, Laplace-Beltrami operator in $\mathcal{H} := \mathsf{L}^2(M, \mathrm{dv})$, essentially self-adjoint on $C_c^\infty(M)$ [Gaffney 51/Cordes 72].
- $H_0 = P^2 \otimes 1 + 1 \otimes \Delta_\Sigma$ in $\mathcal{H}_0 = \mathsf{L}^2(\mathbb{R}, \mathrm{d}x) \otimes \mathsf{L}^2(\Sigma, \mathrm{ds})$ as before.
- $\mathrm{j} \in C^\infty\big(\mathbb{R}; [0,1]\big)$ satisfies $\mathrm{j}(x) := \begin{cases} 1 & \text{if } x \geq 2 \\ 0 & \text{if } x \leq 1 \end{cases}$, and

$$J : \mathcal{H}_0 \to \mathcal{H}, \quad \varphi \mapsto \chi_\infty \sqrt{\frac{\iota^*(1 \otimes \mathfrak{h})}{\mathfrak{g}}}\, \iota^*\big((\mathrm{j} \otimes 1)\varphi\big),$$

with χ_∞ the characteristic function for M_∞, $\mathfrak{g} := \sqrt{\det(g_{jk})}$ and $\mathfrak{h} := \sqrt{\det(h_{jk})}$.

$\rightsquigarrow$ $\|J\varphi\|_{\mathcal{H}} = \|\varphi\|_{\mathcal{H}_0}$ for $\varphi \in \mathcal{H}_0$ with $\mathrm{supp}(\varphi) \subset (2, \infty) \times \Sigma$.

Assumption (Short-range decay of the metric)

There exists $\varepsilon > 0$ *such that for each* $\alpha \in \mathbb{N}^{n+1}$ *and each* $j, k \in \{1, \ldots, n+1\}$:

$$\left|\partial^\alpha \big((\iota^{-1})^* g_{jk} - (1 \oplus h)_{jk}\big)(x, \omega)\right| \leq c_\alpha \langle x \rangle^{-1-\varepsilon}$$

for some constant $c_\alpha \geq 0$ *and for all* $x > 0$ *and* $\omega \in \Sigma$*.*

(On M_∞, the pullback of g converges to the product metric $1 \oplus h$ on $(0, \infty) \times \Sigma$ with rate $\langle x \rangle^{-1-\varepsilon}$.)

$\rightsquigarrow$ compacity conditions between H, H_0, A_0 and $J \ldots$

We take $A := JA_0J^*$ as in the Theorem, but how can we show that A is essentially self-adjoint on $\mathscr{D} := C_c^\infty(M)$?

A is the generator of the strongly continuous unitary group

$$U_t\,\psi := J_t^{1/2} F_t^* \psi, \quad t \in \mathbb{R},\ \psi \in C_c^\infty(M),$$

where for $p \in M$

$$\frac{\mathrm{d}}{\mathrm{d}t} F_t(p) := X_{F_t(p)},$$

$$X := \chi_\infty\, \iota^* \big(\mathrm{j}^2\, \mathrm{id}_{\mathbb{R}} \otimes 1\big)\, (\iota^{-1})_* \left(\frac{\partial}{\partial x}\right),$$

$$J_\tau(p) := \big(\det{}_{\mathrm{dv}} F_\tau\big)(p) \quad \text{with} \quad F_\tau^*\, \mathrm{dv} \equiv \big(\det{}_{\mathrm{dv}} F_\tau\big)\, \mathrm{dv}\,.$$

Since $U_t\,\mathscr{D} \subset \mathscr{D}$ for all $t \in \mathbb{R}$, Nelson's Lemma implies that A is essentially self-adjoint on $\mathscr{D}$.

Summing up:

(i) H_0 is of class $C^1(A_0)$ in $\mathcal{H}_0$,

(ii) $A = JA_0J^*$ is essentially self-adjoint on $\mathscr{D} = C_c^\infty(M)$,

(iii) we have the compacity conditions between H, H_0, A_0 and J.

Thus, H has no singular continuous spectrum in $\mathbb{R} \setminus \{\tau_j\}_{j\geq 0}$ and the point spectrum of H in $\mathbb{R} \setminus \{\tau_j\}_{j\geq 0}$ is composed of eigenvalues of finite multiplicity and with no accumulation point.

Thank you !

References

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