Recent developments in the theory of quantum time delay

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1 Two-body scattering in $\mathbb{R}^d$

- Hilbert space $\mathcal{H}$ (typically $\mathcal{H} = L^2(\mathbb{R}^d)$)
- Free Hamiltonian H_0 (typically $H_0 = h(P)$, with $P := -i\nabla$ and $h \in C^1(\mathbb{R}^d;\mathbb{R})$)
- Full Hamiltonian H (typically $H = H_0 + V$)
- Complete wave operators, *i.e.*

$$W_\pm := \text{s-}\lim_{t\to\pm\infty} e^{itH} e^{-itH_0} P_{ac}(H_0)$$

with $\mathrm{Ran}(W_-) = \mathrm{Ran}(W_+) = \mathcal{H}_{ac}(H)$

$\Longrightarrow$ Unitary scattering operator $S := W_+^* W_-$

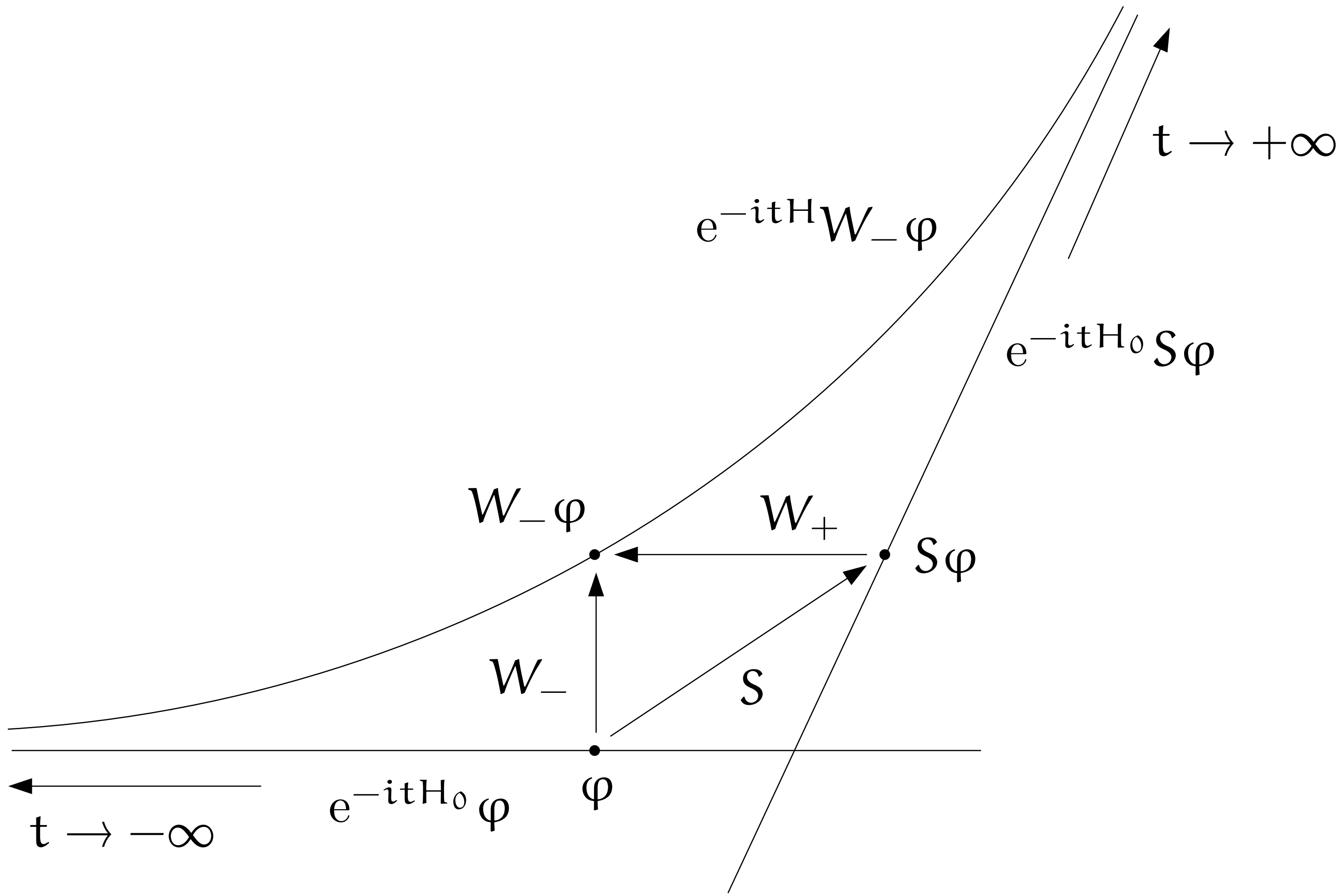


Figure 1: Wave operators $W_\pm$ and scattering operator S

2 Time delay in terms of sojourn times

Take a function $f \in L^\infty(\mathbb{R}^d)$ such that

(i) f decays to 0 at infinity,

(ii) $f = 1$ on a neighbourhood Σ of 0,

(iii) $f(x) = f(-x)$ for almost every $x \in \mathbb{R}^d$ (f is even).

Let $\Phi \equiv (\Phi_1, \ldots, \Phi_d)$ be a family of mutually commuting self-adjoint operators in $\mathcal{H}$.

$\Longrightarrow$ $f(\Phi/r)$, $r > 0$, is approximately the operator of localization in $E^\Phi(r\Sigma)\mathcal{H}$.

Example 2.1. *Let* $\mathcal{H} = L^2(\mathbb{R}^d)$ *and* $Q \equiv (Q_1, \ldots, Q_d)$ *the family of position operators in* $L^2(\mathbb{R}^d)$. *Then* $f(Q/r)$ *is the localization operator in* $\mathcal{B}_r := \{x \in \mathbb{R}^d \mid |x| < r\}$ *if* $f = \chi_{\mathcal{B}_1}$.

Let $\varphi \in \mathcal{H}_{\rm ac}(H_0)$, $\|\varphi\| = 1$, satisfy $\eta(H_0)\varphi = \varphi$ for some appropriate $\eta \in C_c^\infty(\mathbb{R})$.

• Sojourn time of the freely evolving state $e^{-itH_0}\varphi$ in $E^\Phi(r\Sigma)\mathcal{H}$:

$$T_r^0(\varphi) := \int_{\mathbb{R}} dt \left\langle e^{-itH_0}\varphi, f(\Phi/r)e^{-itH_0}\varphi\right\rangle$$

• Sojourn time of the associated scattering state $e^{-itH}W_-\varphi$ in $E^\Phi(r\Sigma)\mathcal{H}$:

$$T_r(\varphi) := \int_{\mathbb{R}} dt \left\langle e^{-itH}W_-\varphi, f(\Phi/r)e^{-itH}W_-\varphi\right\rangle$$

Time delay in $\mathsf{E}^{\Phi}(\mathsf{r}\Sigma)\mathcal{H}$ for the scattering process with incoming state φ:

$$\tau_r^{\mathrm{in}}(\varphi) := \mathsf{T}_r(\varphi) - \mathsf{T}_r^0(\varphi).$$

(definition introduced by Jauch, Misra, and Sinha in the 70's, when $\mathcal{H} = \mathsf{L}^2(\mathbb{R}^d)$, $\Phi = \mathsf{Q}$, $\mathsf{f} = \chi_{\mathcal{B}_1}$ and $\mathsf{H}_0 = -\Delta$)

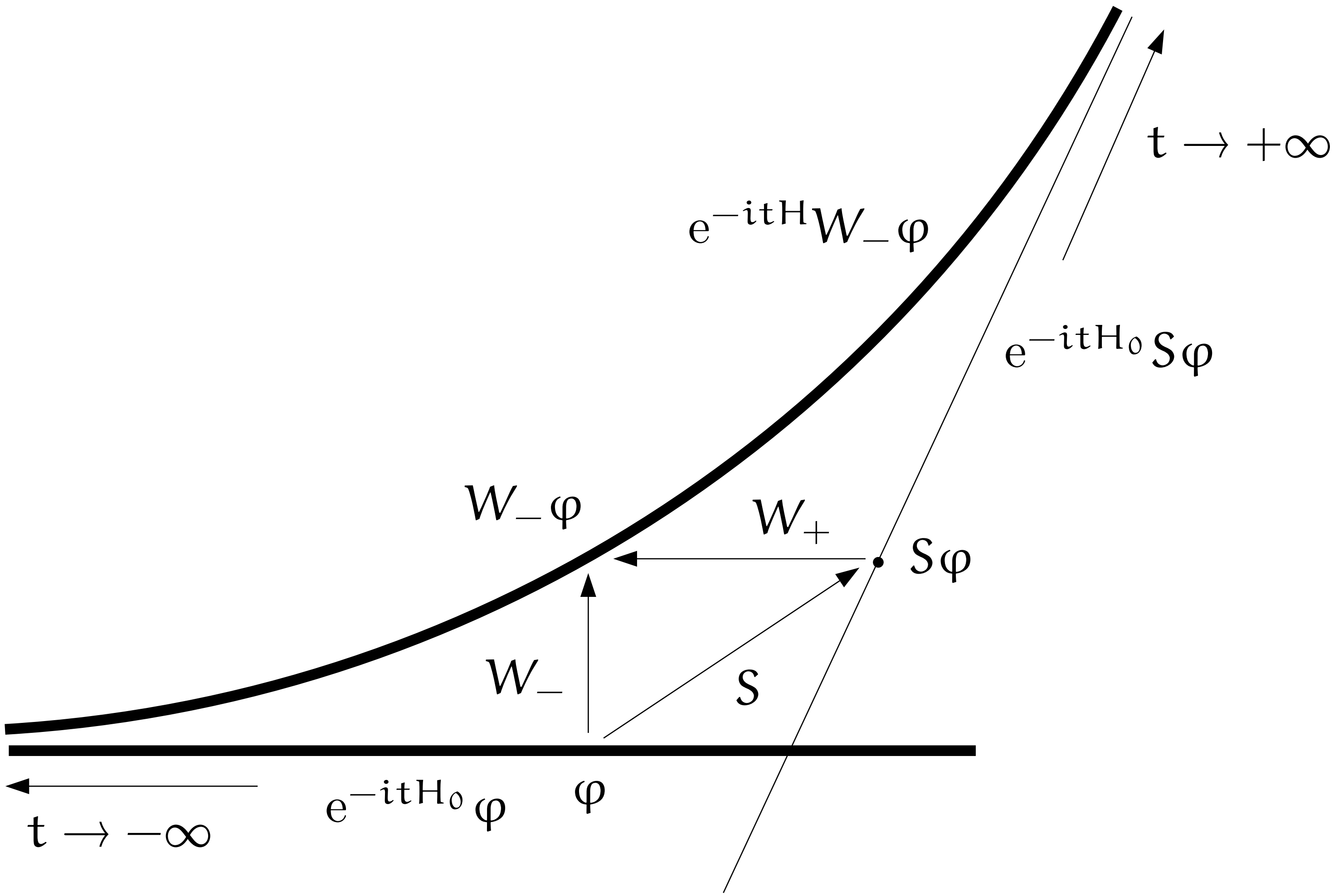


Figure 2: Interpretation of $\tau_r^{\text{in}}(\varphi)$

When $f = \chi_{\mathcal{B}_1}$, $H_0 = -\Delta$, and $H \equiv H_0 + V(Q)$ is short-range, $\tau_r^{\text{in}}(\varphi)$ exists for each $r > 0$, and

$$\begin{aligned}\lim_{r\to\infty} \tau_r^{\text{in}}(\varphi) &= \int_0^\infty d\lambda \left\langle (\mathcal{U}\varphi)(\lambda), -iS(\lambda)^* \frac{dS(\lambda)}{d\lambda}(\mathcal{U}\varphi)(\lambda)\right\rangle_{L^2(\mathbb{S}^{d-1})} \\ &\equiv \langle \varphi, \tau_{\text{E-W}}\varphi\rangle ,\end{aligned}$$

where $\mathcal{U} : \mathcal{H} \to \int_{[0,\infty)}^{\oplus} d\lambda\, L^2(\mathbb{S}^{d-1})$ is the spectral transformation for H_0 and $\{S(\lambda)\}_{\lambda\geq 0}$ the scattering matrix for the pair (H_0, H).

This formula expresses the identity of global time delay (defined in terms of sojourn times) and Eisenbud-Wigner time delay.

(Amrein, Cibils, Jensen, Martin, 80's and 90's)

3 Symmetrised time delay

Alternate (symmetrised) definition:

$$\tau_r(\varphi) := T_r(\varphi) - \frac{1}{2}\left[T_r^0(\varphi) + T_r^0(S\varphi)\right]$$

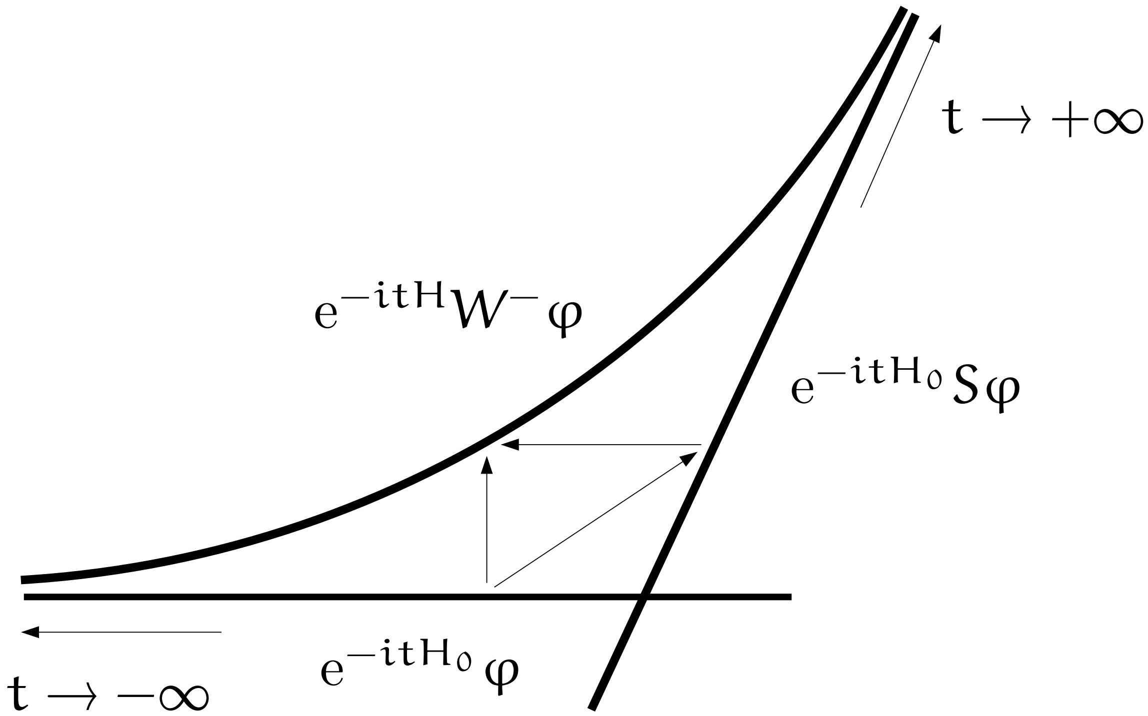


Figure 3: Interpretation of $\tau_r(\varphi)$

For multichannel-type scattering processes, only the symmetrised time delay exists.

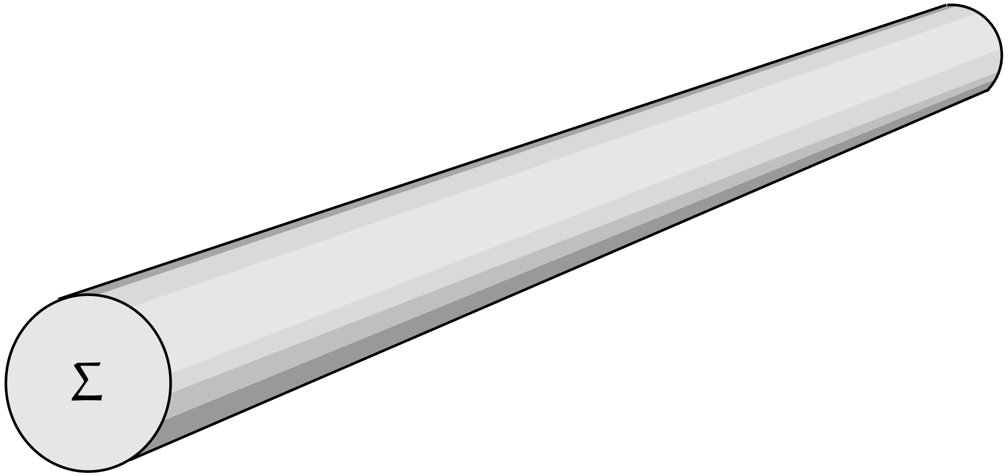
Some examples where the existence of symmetrised time delay has been established (for $f(\Phi/r) = \chi_{\mathcal{B}_1}(Q/r)$):

- Scattering for dissipative interactions (Martin 75),
- "N-body scattering" (Bollé-Osborn 79),
- Scattering in quantum waveguides (T06),
- Scattering for one-dimensionnal anisotropic potentials (Amrein-Jacquet 07).

Consider the scattering pair

$$H_0 = -\Delta_D \quad \text{and} \quad H = H_0 + V(Q) \quad \text{in} \quad \mathcal{H} = L^2(\Sigma \times \mathbb{R}),$$

with $f(\Phi/r) = \chi_{\Sigma\times[-r,r]}(Q) \equiv 1 \otimes \chi_{[-r,r]}(Q_{\mathbb{R}})$.



In such a case, we have

$$\lim_{r\to\infty} \tau_r(\varphi) = \langle \varphi, \tau_{\text{E-W}}\varphi \rangle$$

for a large class of short-range potentials $V(Q)$.

(but $\lim_{r\to\infty} \tau_r^{\text{in}}(\varphi)$ does not exist!)

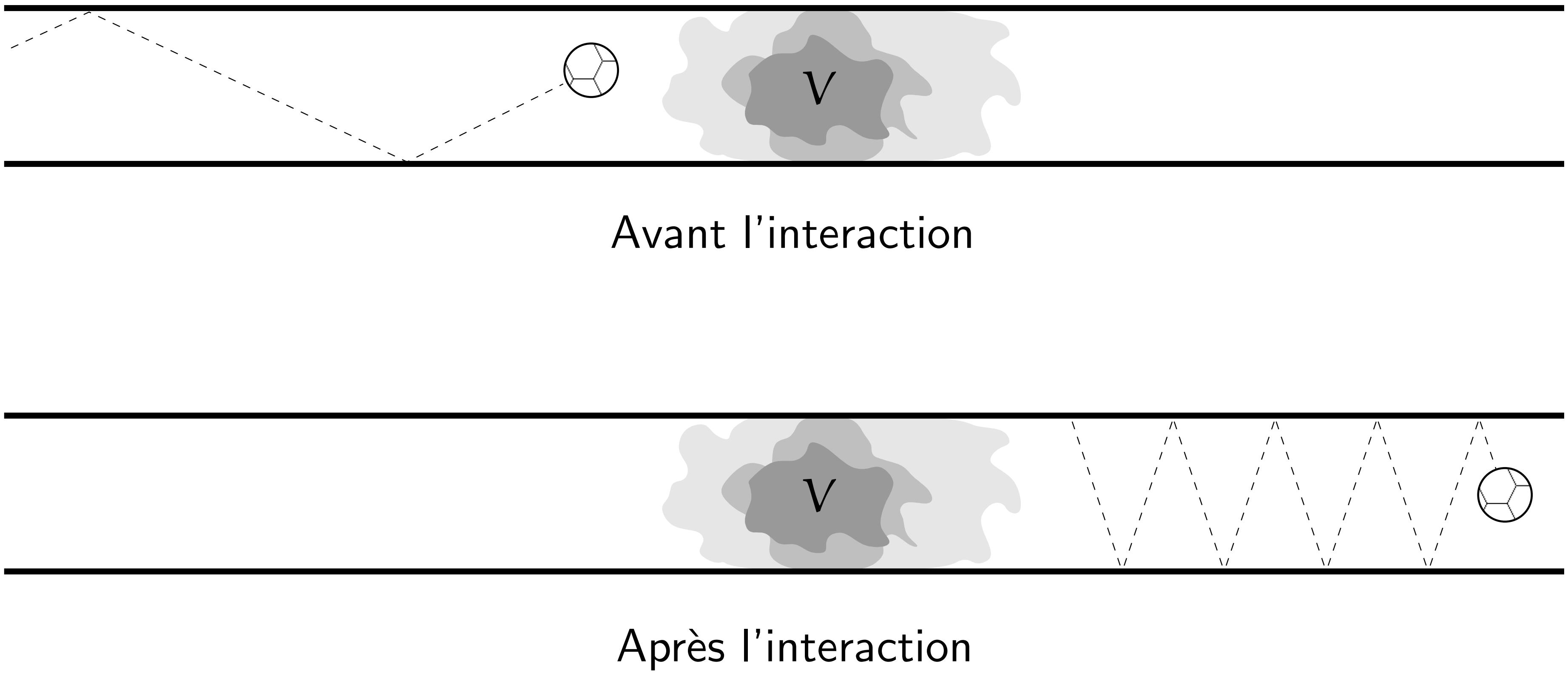


Figure 4: Classical scattering in a waveguide: the usual time delay does not exist because the longitudinal velocities before and after interaction are not comparable

4 Dispersive Hamiltonians

Consider now $H_0 = h(P)$ and $\Phi = Q$ in $\mathcal{H} = L^2(\mathbb{R}^d)$.

Assumption (hypoelliptic-type): The function $h : \mathbb{R}^d \to \mathbb{R}$ is of class C^m for some $m \geq 3$, and satisfies the following conditions:

(i) $|h(x)| \to \infty$ as $|x| \to \infty$.

(ii) $\sum_{|\alpha|\leq m} |(\partial^\alpha h)(x)| \leq \text{Const.}\,(1 + |h(x)|)$.

(iii) $\sum_{|\alpha|=m} |(\partial^\alpha h)(x)| \leq \text{Const.}$

Commutators methods $\Longrightarrow$ H_0 is purely a.c. in $\mathbb{R} \setminus \kappa(H_0)$, with $\kappa(H_0) := h\big[(\nabla h)^{-1}(\{0\})\big]$.

Example 4.1. h *can be an elliptic symbol of degree* $s > 0$.

5 Averaged localization functions

Take a function $f \in L^\infty(\mathbb{R}^d)$ decaying to 0 fast enough at infinity such that $f = 1$ on a neighbourhood of 0.

Then the function $R_f : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$ given by

$$R_f(x) := \int_0^{+\infty} \frac{d\mu}{\mu} \left[f(\mu x) - \chi_{[0,1]}(\mu) \right]$$

is well-defined.

Example 5.1. *If* f *is radial,* i.e. $f(x) = f_0(|x|)$, *then*

$$R_f(x) = R_{f_0}(1) - \ln |x|,$$

and

$$(\nabla R_f)(x) = -x^{-2} x.$$

Theorem 5.2. *Suppose that* $f \in \mathcal{S}(\mathbb{R}^d)$ *is even and satisfies* $f = 1$ *on a neighbourhood of* 0. *Let* h *be as above. Then we have for appropriate* $\varphi \in \mathcal{H}_{\mathrm{ac}}(H_0)$

$$\lim_{r\to\infty}\int_0^\infty \mathrm{d}t\,\big\langle\varphi,\big[\mathrm{e}^{-ith(P)}f(Q/r)\mathrm{e}^{ith(P)}-\mathrm{e}^{ith(P)}f(Q/r)\mathrm{e}^{-ith(P)}\big]\varphi\big\rangle$$
$$=\langle\varphi, T_f\varphi\rangle,$$

where

$$T_f := -\tfrac{1}{2}\big[Q\cdot(\nabla R_f)\big((\nabla h)(P)\big)+(\nabla R_f)\big((\nabla h)(P)\big)\cdot Q\big].$$

Remark 5.3. *If* f *is radial,* T_f *reduces to the operator*

$$T := \frac{1}{2}\Big(Q \cdot \frac{(\nabla h)(P)}{(\nabla h)(P)^2} + \frac{(\nabla h)(P)}{(\nabla h)(P)^2} \cdot Q\Big).$$

In a suitable sense, one has

$$[T, e^{ith(P)}] = -te^{ith(P)} \quad \Longrightarrow \quad T = i\,\frac{d}{dh(P)}\,.$$

Hence, if f is radial and H_0 is purely a.c., the theorem gives

$$\lim_{r\to\infty}\int_0^\infty dt\,\big\langle \varphi, \big[e^{-ith(P)}f(Q/r)e^{ith(P)} - e^{ith(P)}f(Q/r)e^{-ith(P)}\big]\varphi\big\rangle$$

$$= \int_{\sigma(H_0)} d\lambda\,\Big\langle (\mathcal{U}\varphi)(\lambda), i\,\frac{d(\mathcal{U}\varphi)}{d\lambda}(\lambda)\Big\rangle_{\mathcal{H}_\lambda},$$

where $\mathcal{U} : \mathcal{H} \to \int_{\sigma(H_0)} d\lambda\, \mathcal{H}_\lambda$ is a spectral transformation for $H_0 = h(P)$.

6 Existence of symmetrised time delay

Let H be any selfadjoint perturbation of $\mathsf{H}_0 = \mathsf{h}(\mathsf{P})$ satisfying the following condition.

Assumption $W_\pm$: The wave operators $W_\pm$ exist and are complete, and any operator $\mathsf{T} \in \mathcal{B}\big(\mathcal{D}(\langle \mathsf{Q}\rangle^{-\rho}), \mathcal{H}\big)$, with $\rho > \frac{1}{2}$, is locally H-smooth on $\mathbb{R} \setminus \{\kappa(\mathsf{H}_0) \cup \sigma_{\rm pp}(\mathsf{H})\}$.

Theorem 6.1. *Let* $f \in \mathcal{S}(\mathbb{R}^d)$ *be an even function such that* $f = 1$ *on a neighbourhood of* 0*. Let* h *be as above. Suppose that Assumption* $W_\pm$ *holds. Then, we have for appropriate* $\varphi \in \mathcal{H}_{\rm ac}(H_0)$

$$\lim_{r\to\infty} \tau_r(\varphi) = -\langle \varphi, S^*[T_f, S]\varphi\rangle\,.$$

Remark 6.2. *If* f *is radial and* H_0 *is purely a.c., we get the identity of symmetrised time delay and Eisenbud-Wigner time delay for dispersive Hamiltonians* $H_0 = h(P)$*:*

$$\lim_{r\to\infty} \tau_r(\varphi) = \int_{\sigma(H_0)} d\lambda \left\langle (\mathcal{U}\varphi)(\lambda), -iS(\lambda)^* \frac{dS(\lambda)}{d\lambda}\,(\mathcal{U}\varphi)(\lambda)\right\rangle_{\mathcal{H}_\lambda}.$$

7 Equality of symmetrised time delay and usual time delay

Let $F_f : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$ be defined by

$$F_f(x) := \int_{\mathbb{R}} d\mu \, f(\mu x).$$

If $p \in \mathbb{R}^d$ and f is real, the number $F_f(p) \equiv \int_{\mathbb{R}} dt \, f(tp)$ can be seen as the sojourn time in the region defined by the localization function f of a free classical particle moving along the trajectory $\mathbb{R} \ni t \mapsto x(t) := tp$.

$F_f\big((\nabla h)(P)\big)$ is a "quantum analog" of $F_f(p)$, since $(\nabla h)(P)$ is the quantum velocity operator.

Theorem 7.1. *Let* $f \in \mathcal{S}(\mathbb{R}^d)$ *be even. Let* h *be as above. Suppose that Assumption* $W_\pm$ *holds. Assume that*

$$\left[F_f\big((\nabla h)(P)\big), S\right] = 0. \tag{1}$$

Then, we have for appropriate $\varphi \in \mathcal{H}_{\mathrm{ac}}(H_0)$

$$\lim_{r\to\infty}\left[\tau_r^{\mathrm{in}}(\varphi) - \tau_r(\varphi)\right] = 0.$$

Example 7.2. *In the case of waveguides, the (natural) velocity operator*

$$1 \otimes P_{\mathbb{R}} \equiv i\big[-\Delta_D^\Sigma \otimes 1 + 1 \otimes P_{\mathbb{R}}^2, 1 \otimes Q_{\mathbb{R}}\big] \equiv i\big[-\Delta_D, 1 \otimes Q_{\mathbb{R}}\big]$$

does not commute with the scattering operator S.

Sketch of the proof:

Using the change of variables $\mu := t/r$, $\nu := 1/r$, and the parity of f, one gets

$$
\begin{aligned}
&2 \lim_{r\to\infty} \left[\tau_r(\varphi) - \tau_r^{\mathrm{in}}(\varphi)\right] \\
&= \lim_{r\to\infty} \int_{\mathbb{R}} \mathrm{d}t \left\langle \varphi, S^*[\mathrm{e}^{ith(P)} f(Q/r) \mathrm{e}^{-ith(P)}, S]\varphi \right\rangle \\
&= \lim_{r\to\infty} \int_{\mathbb{R}} \mathrm{d}t \left\langle \varphi, S^*[\mathrm{e}^{ith(P)} f(Q/r) \mathrm{e}^{-ith(P)}, S]\varphi \right\rangle \\
&\qquad - \left\langle \varphi, S^*\left[F_f\big((\nabla h)(P)\big), S\right]\varphi \right\rangle \\
&= \lim_{\nu\searrow 0} \int_{\mathbb{R}} \mathrm{d}\mu \left\langle \varphi, S^*\left[\tfrac{1}{\nu}\left\{f[\nu Q + \mu(\nabla h)(P)] - f\big(\mu(\nabla h)(P)\big)\right\}, S\right]\varphi \right\rangle \\
&= \int_{\mathbb{R}} \mathrm{d}\mu \left\langle \varphi, S^*\left[Q \cdot (\nabla f)\big(\mu(\nabla h)(P)\big), S\right]\varphi \right\rangle \\
&= 0.
\end{aligned}
$$

There are two simple situations where condition (1) is satisfied:

1. If h is a polynomial of degree 1, *i.e.* $h(x) = v_0 + v \cdot x$ for some $v_0 \in \mathbb{R}$, $v \in \mathbb{R}^d \setminus \{0\}$. Then $F_f\big((\nabla h)(P)\big)$ reduces to the scalar $F_f(v)$, and thus it commutes with S.

(This covers the case of the Friedrichs Hamiltonian.)

2. If f and h are radial, *i.e.* $f(x) = f_0(|x|)$ and $h(x) = h_0(|x|)$ with $h_0' \geq 0$ on $\mathbb{R}_+$. Then $F_f\big((\nabla h)(P)\big) = F_{f_0}\big(h_0'(|P|)\big)$ is diagonalizable in the spectral representation of $H_0 \equiv h(P)$. So it commutes with S.

(This covers the Schrödinger case $h_0(\rho) = \rho^2$, the square-root Klein-Gordon case $h_0(\rho) = \sqrt{1 + \rho^2}$, and others.)

8 Existence of usual time delay

Theorem 8.1. *Let* $f \in \mathcal{S}(\mathbb{R}^d)$ *be an even function such that* $f = 1$ *on a neighbourhood of* 0*. Let* h *be as above. Suppose that Assumption* $W_\pm$ *holds. Assume that*

$$\big[F_f\big((\nabla h)(P)\big), S\big] = 0.$$

Then, we have for appropriate $\varphi \in \mathcal{H}_{\mathrm{ac}}(H_0)$

$$\lim_{r\to\infty} \tau_r^{\mathrm{in}}(\varphi) = \lim_{r\to\infty} \tau_r(\varphi) = -\left\langle \varphi, S^*[T_f, S]\varphi \right\rangle .$$

9 Further developments

- The case when $\mathcal{H}$ is an abstract Hilbert space, $(\mathsf{H}_0, \mathsf{H})$ an abstract scattering pair and $\Phi \equiv (\Phi_1, \ldots, \Phi_d)$ a general family of mutually commuting self-adjoint operators.

- Time delay in terms of sojourn times in Hamiltonian mechanics.

10 Some references

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