Commutator criteria for strong mixing

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Strong mixing

Example (Discrete group of unitary operators)

If U is a unitary operator in a Hilbert space $\mathcal{H}$,

$$U_n := U^n, \quad n \in \mathbb{Z},$$

defines a discrete 1*-parameter group of unitary operators.*

Example (Continuous group of unitary operators)

If H is a self-adjoint operator in a Hilbert space $\mathcal{H}$, then

$$U_t := \mathrm{e}^{-itH}, \quad t \in \mathbb{R},$$

defines a strongly continuous 1*-parameter group of unitary operators.*

Example (Koopman operator)

If $T : X \to X$ is an automorphism of a probability space (X, μ), then the Koopman operator

$$U_T : \mathrm{L}^2(X, \mu) \to \mathrm{L}^2(X, \mu), \quad \varphi \mapsto \varphi \circ T,$$

is a unitary operator.

Ergodicity, weak mixing and strong mixing of an automorphism $T : X \to X$ are expressible in terms of the Koopman operator U_T :

- T is ergodic iff 1 is a simple eigenvalue of U_T.
- T is weakly mixing iff U_T has purely continuous spectrum in $\{\mathbb{C} \cdot \mathbf{1}\}^\perp$.
- T is strongly mixing iff

$$\lim_{N\to\infty} \langle \varphi, (U_T)^N \varphi \rangle = 0 \quad \text{for all } \varphi \in \{\mathbb{C} \cdot \mathbf{1}\}^\perp.$$

a.c. spectrum in $\{\mathbb{C} \cdot \mathbf{1}\}^\perp$ $\Rightarrow$ strong mixing $\Rightarrow$ weak mixing $\Rightarrow$ ergodicity

Commutators

- $\mathcal{H}$, arbitrary Hilbert space with norm $\|\cdot\|$ and scalar product $\langle\cdot,\cdot\rangle$
- $\mathscr{B}(\mathcal{H})$, bounded linear operators on $\mathcal{H}$
- A, self-adjoint operator in $\mathcal{H}$ with domain $\mathcal{D}(A)$

Definition

An operator $S \in \mathscr{B}(\mathcal{H})$ satisfies $S \in C^k(A)$ if

$$\mathbb{R} \ni t \mapsto \mathrm{e}^{-itA} S \, \mathrm{e}^{itA} \in \mathscr{B}(\mathcal{H})$$

is strongly of class C^k.

$S \in C^1(A)$ if and only if

$$\big|\langle A\varphi, S\varphi\rangle - \langle \varphi, SA\varphi\rangle\big| \leq \text{Const.} \, \|\varphi\|^2 \quad \text{for all } \varphi \in \mathcal{D}(A).$$

The operator corresponding to the continuous extension of the quadratic form is denoted by $[S, A]$, and one has

$$[iS, A] = \text{s-} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{e}^{-itA} S \, \mathrm{e}^{itA} \in \mathscr{B}(\mathcal{H}).$$

Definition

A self-adjoint operator H in $\mathcal{H}$ is of class $C^k(A)$ if $(H-z)^{-1} \in C^k(A)$ for some $z \in \mathbb{C} \setminus \sigma(H)$.

If H is of class $C^1(A)$, then

$$\left[A,(H-z)^{-1}\right] = (H-z)^{-1}\,[H,A]\,(H-z)^{-1},$$

with $[H,A] \in \mathscr{B}\big(\mathcal{D}(H), \mathcal{D}(H)^*\big)$ the operator corresponding to the continuous extension to $\mathcal{D}(H)$ of the quadratic form

$$\mathcal{D}(H) \cap \mathcal{D}(A) \ni \varphi \mapsto \langle H\varphi, A\varphi\rangle - \langle A\varphi, H\varphi\rangle \in \mathbb{C}.$$

Discrete groups

Theorem (Strong mixing for discrete groups)

Let U be a unitary operator in $\mathcal{H}$ and let A be a self-adjoint operator in $\mathcal{H}$ with $U \in C^1(A)$. Assume that the strong limit

$$D := \operatorname*{s\text{-}lim}_{N\to\infty} \frac{1}{N}\left[A, U^N\right] U^{-N} = \operatorname*{s\text{-}lim}_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} U^n\left([A,U]U^{-1}\right)U^{-n}$$

exists. Then,

(a) $\lim_{N\to\infty}\left\langle \varphi, U^N\psi\right\rangle = 0$ *for each* $\varphi \in \ker(D)^\perp$ *and* $\psi \in \mathcal{H}$,

(b) $U|_{\ker(D)^\perp}$ *has purely continuous spectrum.*

- D is bounded and self-adjoint because it is the strong limit of bounded self-adjoint operators.
- $DU^n = U^n D$ for each $n \in \mathbb{Z}$. So, $\ker(D)^\perp$ is a reducing subspace for U, and $U|_{\ker(D)^\perp}$ is a unitary operator.
- Point (b) is a simple consequence of point (a).

Sketch of the proof of (a).

Let $\varphi = D\widetilde{\varphi} \in D\mathcal{D}(A)$, $\psi \in \mathcal{D}(A)$, $N \in \mathbb{N}^*$, and

$$D_N := \tfrac{1}{N}\left[A, U^N\right]U^{-N}.$$

Since $U^N, U^{-N} \in C^1(A)$, we have $U^N\psi, U^{-N}\widetilde{\varphi} \in \mathcal{D}(A)$. Thus,

$$\begin{aligned}
&\left|\langle \varphi, U^N\psi\rangle\right| \\
&= \left|\langle (D - D_N)\widetilde{\varphi}, U^N\psi\rangle + \langle D_N\widetilde{\varphi}, U^N\psi\rangle\right| \\
&\le \left\|(D - D_N)\widetilde{\varphi}\right\| \|\psi\| + \frac{1}{N}\left|\langle \left[A, U^N\right]U^{-N}\widetilde{\varphi}, U^N\psi\rangle\right| \\
&\le \left\|(D - D_N)\widetilde{\varphi}\right\| \|\psi\| + \frac{1}{N}\left|\langle A\widetilde{\varphi}, U^N\psi\rangle\right| + \frac{1}{N}\left|\langle U^N A U^{-N}\widetilde{\varphi}, U^N\psi\rangle\right| \\
&\le \left\|(D - D_N)\widetilde{\varphi}\right\| \|\psi\| + \frac{1}{N}\left\|A\widetilde{\varphi}\right\| \|\psi\| + \frac{1}{N}\left\|\widetilde{\varphi}\right\| \|A\psi\|.
\end{aligned}$$

Since $D = \text{s-lim}_N D_N$, we get $\lim_N \langle \varphi, U^N\psi\rangle = 0$, and the claim follows from density arguments. □

Remark

If $\sum_{N\geq 1} \|(D - D_N)\varphi\|^2 < \infty$ *for suitable* $\varphi \in \mathcal{H}$*, then*

$$\sum_{N\geq 1} \left|\langle \varphi, U^N \varphi\rangle\right|^2 < \infty \quad \textit{for all } \varphi \in \ker(D)^\perp,$$

and $U|_{\ker(D)^\perp}$ *has purely a.c. spectrum.*

Example (Cocycles with values in compact Lie groups)

. . .

Continuous groups

Theorem (Strong mixing for continuous groups)

Let H and A be self-adjoint operators in $\mathcal{H}$ with $(H-i)^{-1} \in C^1(A)$. Assume that

$$D := \operatorname*{s-lim}_{t\to\infty} \frac{1}{t}\int_0^t \mathrm{d}s\, \mathrm{e}^{isH}(H+i)^{-1}[iH,A](H-i)^{-1}\,\mathrm{e}^{-isH}$$

exists. Then,

(a) $\lim_{t\to\infty} \langle \varphi, \mathrm{e}^{-itH}\,\psi\rangle = 0$ *for each* $\varphi \in \mathcal{H}$ *and* $\psi \in \ker(D)^\perp$,

(b) $H|_{\ker(D)^\perp}$ *has purely continuous spectrum.*

- The proof is similar to the one for unitary operators (just more domain issues because both H and A are unbounded).
- The results in the unitary case (the group $(\mathbb{Z}, +)$) and in the self-ajoint case (the group $(\mathbb{R}, +)$) are particular cases of a more general criterion for the strong mixing property of unitary representations of topological groups.

Example (Canonical commutation relation)

Assume that $(H-i)^{-1} \in C^1(A)$ *with* $[iH, A] = 1$*. Then, for all* $t > 0$

$$D_t := \frac{1}{t}\int_0^t \mathrm{d}s\; \mathrm{e}^{isH}(H+i)^{-1}\,[iH, A]\,(H-i)^{-1}\,\mathrm{e}^{-isH} = (H^2+1)^{-1} = D$$

and $\ker(D) = \{0\}$*. So, the theorem implies that* H *has purely a.c. spectrum. In fact, we have in this case the Weyl commutation relation*

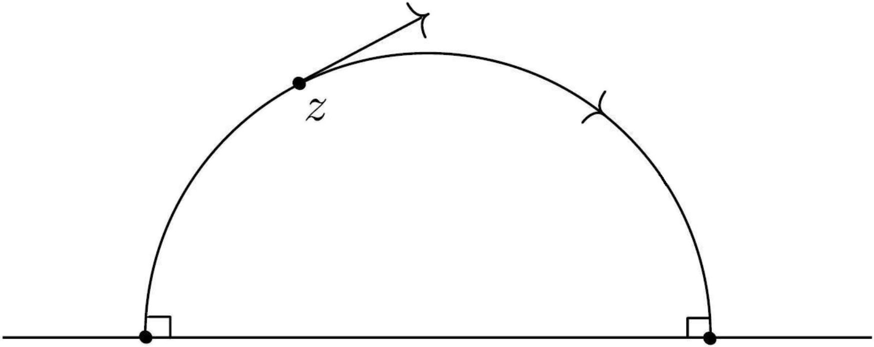
$$\mathrm{e}^{-itA}\,\mathrm{e}^{isH}\,\mathrm{e}^{itA} = \mathrm{e}^{ist}\,\mathrm{e}^{isH}, \quad s, t \in \mathbb{R}.$$

Thus, Stone-von Neumann theorem implies that H *has Lebesgue spectrum with uniform multiplicity.*

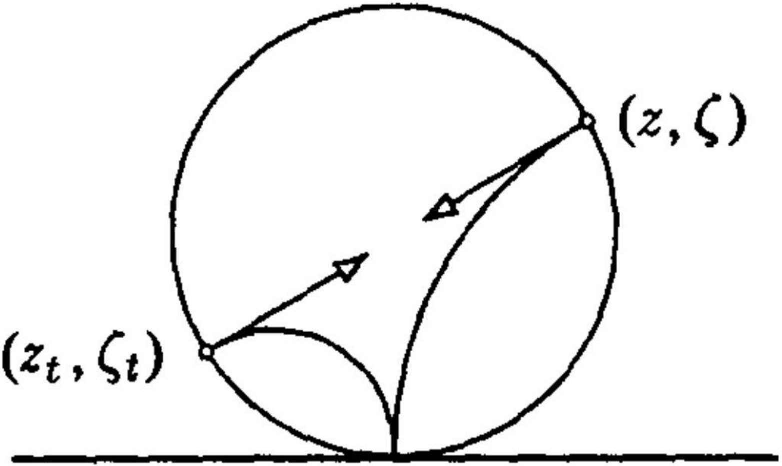
Example (Time changes of horocycle flows).

- Σ, compact Riemannian surface of constant negative curvature
- $M := T^1\Sigma$, unit tangent bundle of Σ
  (M is a compact 3-manifold with probability measure μ,
  $M \simeq \Gamma \setminus \mathrm{PSL}(2;\mathbb{R})$ for some cocompact lattice Γ in $\mathrm{PSL}(2;\mathbb{R})$)
- $F_h := \{F_{h,t}\}_{t\in\mathbb{R}}$, horocycle flow on M
- $F_g := \{F_{g,t}\}_{t\in\mathbb{R}}$, geodesic flow on M

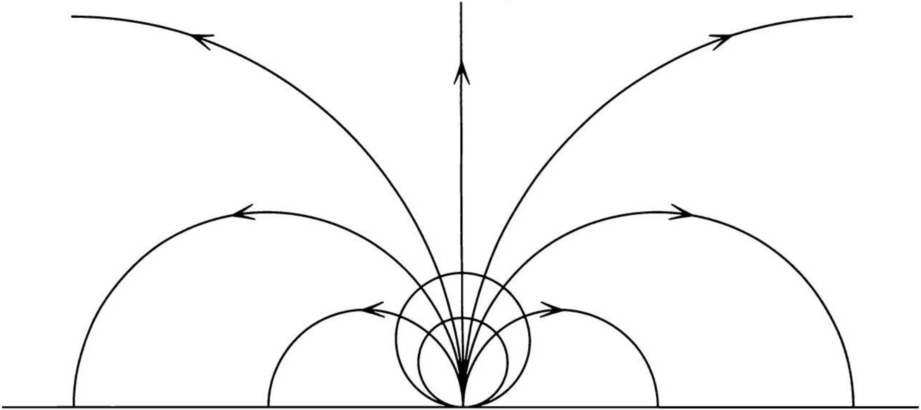
F_h and F_g are one-parameter groups of diffeomorphisms preserving the measure μ.



Geodesic flow in the Poincaré half plane



Positive horocycle flow in the Poincaré half plane
(from Bekka/Mayer's book)



Geodesics and horocycles in the Poincaré half plane
(from Hasselblatt/Katok's book)

Each flow has an essentially self-adjoint generator

$$H_j\varphi := -iX_j\varphi, \quad \varphi \in C^\infty(M) \subset \mathrm{L}^2(M,\mu),$$

with X_j the vector field of F_h or F_g. H_h is of class $C^1(H_g)$ with

$$\big[iH_h, H_g\big] = H_h.$$

A C^1-time change of X_h is a vector field fX_h with $f \in C^1\big(M;(0,\infty)\big)$.
fX_h has a complete flow $\widetilde{F}_h := \{\widetilde{F}_{h,t}\}_{t\in\mathbb{R}}$ with generator $H := fH_h$ essentially self-adjoint on $C^1(M) \subset \mathcal{H} := \mathrm{L}^2(M, f^{-1}\mu)$.

$A := f^{1/2} H_g f^{-1/2}$ is self-adjoint in $\mathcal{H}$, and $(H-i)^{-1} \in C^1(A)$ with

$$(H+i)^{-1}[iH, A](H-i)^{-1} = (H+i)^{-1}\big(H\xi + \xi H\big)(H-i)^{-1}$$

and

$$\xi := \frac{1}{2} - \frac{1}{2} f^{-1} X_g(f).$$

So,

$$\begin{aligned} D_t &= \frac{1}{t}\int_0^t \mathrm{d}s\ \mathrm{e}^{isH}(H+i)^{-1}[iH, A](H-i)^{-1}\,\mathrm{e}^{-isH} \\ &= (H+i)^{-1}\big(H\xi_t + \xi_t H\big)(H-i)^{-1} \end{aligned}$$

with

$$\xi_t := \frac{1}{t}\int_0^t \mathrm{d}s\ \mathrm{e}^{isH}\,\xi\,\mathrm{e}^{-isH} = \frac{1}{t}\int_0^t \mathrm{d}s\,\big(\xi \circ \widetilde{F}_{h,-s}\big).$$

Since F_h is uniquely ergodic with respect to μ, $\widetilde{F}_h$ is uniquely ergodic with respect to $\widetilde{\mu} := \frac{f^{-1}\mu}{\int_M f^{-1}\mathrm{d}\mu}$. Thus,

$$\lim_{t\to\infty} \xi_t = \frac{1}{2} - \frac{1}{2}\int_M \mathrm{d}\widetilde{\mu}\, f^{-1} X_g(f) = \frac{1}{2} + \frac{1}{2\int_M f^{-1}\mathrm{d}\mu}\int_M \mathrm{d}\mu\, X_g\big(f^{-1}\big) = \frac{1}{2}$$

uniformly on M, and

$$D := \operatorname*{s-lim}_{t\to\infty} D_t = (H+i)^{-1}\left(H\cdot\tfrac{1}{2} + \tfrac{1}{2}\cdot H\right)(H-i)^{-1} = H\left(H^2+1\right)^{-1}.$$

So, $\ker(D) = \ker(H)$, and the theorem implies that

$$\lim_{t\to\infty}\left\langle \varphi, \mathrm{e}^{-itH}\psi\right\rangle = 0 \quad \text{for all } \varphi\in\mathcal{H} \text{ and } \psi\in\ker(H)^{\perp}.$$

Therefore, C^1-time changes of horocycle flows are strongly mixing.

Thank you !

References

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