Spectral analysis for convolution operators on locally compact groups

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PUC, May 2009

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1 Motivation: discrete Laplacian on $\mathbb{Z}$

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The discrete Laplacian on $\mathbb{Z}$ is the bounded selfadjoint convolution operator given by

$$(-\Delta_{\mathbb{Z}} f)(x) := -\sum_{|y-x|=1} [f(y)-f(x)] \equiv \Big[(2\delta_0-\delta_{-1}-\delta_1)*f\Big](x), \quad f \in \ell^2(\mathbb{Z}).$$

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It satisfies

$$\sigma(-\Delta_{\mathbb{Z}}) = [0,2] \qquad \text{and} \qquad -\Delta_{\mathbb{Z}} = -2(\operatorname{Re} T - 1),$$

where

$$(Tf)(x) := f(x+1), \quad T^*T = T^*T = 1.$$

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$$(\Phi f)(x) := xf(x),$$

and the discrete momentum operator (finite difference operator)

$$K := i[-\Delta_{\mathbb{Z}}, \Phi] = 2 \operatorname{Im} T.$$

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Then

$$A := \tfrac{1}{2}(\Phi K + K \Phi)$$

is a discrete version of the dilations generator of quantum mechanics:

$$D := \tfrac{1}{2}(Q \cdot P + P \cdot Q) \equiv \tfrac{1}{2}\big(Q \cdot i\big[\tfrac{P^2}{2}, Q\big] + i\big[\tfrac{P^2}{2}, Q\big] \cdot Q\big).$$

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$\Longrightarrow$ A is a good candidate as a conjugate operator for $-\Delta_{\mathbb{Z}}$.

Indeed one has

$$\mathrm{i}[-\Delta_{\mathbb{Z}}, A] = \mathsf{K}^2 = -\Delta_{\mathbb{Z}}(4 + \Delta_{\mathbb{Z}}) \quad \text{and} \quad \mathrm{i}[\mathsf{K}^2, A] \in \mathcal{B}[\ell^2(\mathbb{Z})].$$

Indeed one has

$$i[-\Delta_{\mathbb{Z}}, A] = K^2 = -\Delta_{\mathbb{Z}}(4 + \Delta_{\mathbb{Z}}) \quad \text{and} \quad i[K^2, A] \in \mathcal{B}[\ell^2(\mathbb{Z})].$$

Since the function $\mathbb{R} \ni x \mapsto x(4 - x)$ is strictly positive on $(0, 2] = \sigma(-\Delta_{\mathbb{Z}}) \setminus \{0\}$, Mourre theory implies that the spectrum of $-\Delta_{\mathbb{Z}}$ is purely absolutely continuous on $(0, 2]$.

(Since $\ker(-\Delta_{\mathbb{Z}}) = \{0\}$, the spectrum of $-\Delta_{\mathbb{Z}}$ is even purely absolutely continuous on $[0, 2]$.)

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- $\mathsf{M}(\mathsf{X})$, *complex bounded Radon measures* on X,
- $\mathsf{C}_0(\mathsf{X})$, continuous functions on X decaying to 0 at infinity.

For $\mu, \nu \in \mathsf{M}(\mathsf{X})$ and $f \in \mathcal{H} := \mathsf{L}^2(\mathsf{X}, \mathrm{d}\lambda)$ we set

$$(\mu * f)(x) := \int_{\mathsf{X}} \mathrm{d}\mu(y)\, f(y^{-1}x), \quad x \in \mathsf{X},$$

and

$$\int_{\mathsf{X}} \mathrm{d}(\mu * \nu)(x)\, g(x) := \int_{\mathsf{X}} \int_{\mathsf{X}} \mathrm{d}\mu(x)\mathrm{d}\nu(y)\, g(xy), \quad g \in \mathsf{C}_0(\mathsf{X}).$$

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Given $\varphi : \mathsf{X} \to \mathbb{R}$ (nice enough), $\varphi\mu \in \mathsf{M}(\mathsf{X})$ is associated to the bounded linear functional

$$\mathsf{C}_0(\mathsf{X}) \ni g \mapsto \int_{\mathsf{X}} d\mu(x)\, \varphi(x) g(x) \in \mathbb{C}.$$

The *convolution operator* H_μ, $\mu \in \mathsf{M}(\mathsf{X})$, is defined by

$$(\mathsf{H}_\mu f)(x) := (\mu * f)(x), \quad f \in \mathcal{H},\ x \in \mathsf{X}.$$

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$$(\mathsf{H}_\mu \mathsf{f})(\mathsf{x}) := (\mu * \mathsf{f})(\mathsf{x}), \quad \mathsf{f} \in \mathcal{H},\ \mathsf{x} \in \mathsf{X}.$$

One has

1. $\|\mathsf{H}_\mu\| \leq |\mu|(\mathsf{X})$,
2. $(\mathsf{H}_\mu)^* = \mathsf{H}_{\mu^*}$, where $\mu^*(\mathsf{E}) = \overline{\mu(\mathsf{E}^{-1})}$ (E Borel subset of X).

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$\Longrightarrow$ H_μ is bounded and selfadjoint if $\mu = \mu^* \in \mathsf{M}(\mathsf{X})$.

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Definition 3.1. *Let* $\mu = \mu^* \in \mathsf{M}(\mathsf{X})$. *Then*

(a) $\Phi \in \mathrm{Hom}(\mathsf{X}, \mathbb{R})$ *belongs to* $\mathrm{Hom}^1_\mu(\mathsf{X}, \mathbb{R})$ *if*

$$\Phi\mu, \Phi^2\mu \in \mathsf{M}(\mathsf{X}), \qquad \textit{(decay assumptions)},$$
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(b) $\Phi \in \mathrm{Hom}^1_\mu(\mathsf{X}, \mathbb{R})$ *belongs to* $\mathrm{Hom}^2_\mu(\mathsf{X}, \mathbb{R})$ *if*

$$\Phi^3\mu \in \mathsf{M}(\mathsf{X}), \qquad \textit{(stronger decay assumptions)},$$
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$$\mathcal{H}_{\mathrm{p}}(\mathsf{H}_\mu) \subset \mathcal{K}_\mu^1 \quad \text{and} \quad \mathcal{H}_{\mathrm{sing}}(\mathsf{H}_\mu) \subset \mathcal{K}_\mu^2.$$

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- The spectral structure of H_{μ} is somehow related to the "size" of the space $\mathrm{Hom}(X, \mathbb{R})$?

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   (by analogy to
   $D = \frac{1}{2}(Q \cdot P + P \cdot Q) \equiv \frac{1}{2}\big(Q \cdot i\big[\frac{P^2}{2},Q\big] + i\big[\frac{P^2}{2},Q\big] \cdot Q\big)$ in
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3. One has $i[H_\mu, A] = K^2 \geq 0$, and thus

$$i[H_\mu, A] = K^2 > 0 \quad \text{on } [\ker(K)]^\perp.$$

4. Apply the following Kato-Putnam-type theorem with

$$\mathcal{H}_0 = [\ker(\mathsf{K})]^{\perp}, \quad \mathsf{A}_0 = \mathsf{A} \restriction \mathcal{H}_0, \quad \mathsf{H}_0 = \mathsf{H}_{\mu} \restriction \mathcal{H}_0.$$

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If H_0 is a bounded selfadjoint operator in a Hilbert space $\mathcal{H}_0$, one has:

Definition 4.1. A selfadjoint operator A_0 in $\mathcal{H}_0$ is *weakly conjugate to* H_0 if $i[\mathsf{H}_0, A_0]$ is bounded and $i[\mathsf{H}_0, A_0] > 0$.

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5. Thus

$$\mathcal{H}_{\text{sing}}(\mathsf{H}_\mu) \subset \ker(\mathsf{K}) = \ker(\mathsf{H}_{\Phi\mu}).$$

5 Examples

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Accordingly $\mathcal{H}_\mathrm{p}(\mathsf{H}_\mu)=\mathcal{H}$ *since* H_μ *can be written as a direct sum of compact operators.*

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Example 5.2 (Discrete Laplacian on $\mathbb{Z}$)**.** *For* $\mu=2\delta_0-\delta_{-1}-\delta_1$ *and* $(\Phi f)(x):=xf(x)$*, one has* $\mathsf{K}\equiv-i\mathsf{H}_{\Phi\mu}=2\,\mathrm{Im}\,\mathsf{T}$ *and* $\ker(\mathsf{K})=\{f\in\mathcal{H}\mid f(\cdot+1)=f(\cdot-1)\}=\{0\}$*. Thus*

$$\ker(\mathsf{H}_\mu)=\mathcal{K}_\mu^1=\mathcal{K}_\mu^2=\{0\},$$

and H_μ *is purely absolutely continuous.*

Example 5.3 (Central groups)**.** *Let* X *be central, take a non compact element* $z \in Z(X)$*, and set* $\mu := \delta_z + \delta_{z^{-1}}$*. Choose* $\Phi \in \mathrm{Hom}(X, \mathbb{R})$ *such that* $\Phi(z) = \frac{1}{2}\Phi(z^2) \neq 0$ *(such a* Φ *exists because* X *is central). Then*

$$\mathcal{H}_{\mathrm{sing}}(H_\mu) \subset \ker(H_{\Phi\mu}) = \left\{ f \in \mathcal{H} \mid f(z^{-1}\,\cdot\,) = f(z\,\cdot\,) \right\} = \{0\}.$$

Thus $\mathcal{H}_{\mathrm{ac}}(H_\mu) = \mathcal{H}$*.*

Other examples can be found on $X = S_3 \times \mathbb{Z}$*,* $X = \mathrm{SU}(2) \times \mathbb{R}$*, etc.*

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Example 5.4 (Abelian groups)**.** *Application of the Fourier transform to Theorem 3.2 gives results on multiplication operators.*

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Example 5.5 (Semidirect products). $\mathcal{H}_{\mathrm{ac}}(H_a) \neq \{0\}$ *for appropriate functions* $a = a^* \in L^1(X)$ *on* $X = N \times_\tau G$, *where* N, G *are discrete with* G *abelian and* $\tau : G \to \mathrm{Aut}(N)$ *is a suitable group morphism.*

6 Some references

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