Commutator methods with applications to the spectral analysis of dynamical systems

Rafael Tiedra
(Catholic University of Chile)

Penn State, January 2013

Contents

1 Commutator methods for self-adjoint operators

Commutator methods are a tool for the spectral theory and the scattering theory of self-adjoint operators in Hilbert spaces.

They have been introduced by Éric Mourre in the 80's for the study of Schrödinger operators in $\mathsf{L}^2(\mathbb{R}^d)$ (and further developed by Amrein, Boutet de Monvel, Georgescu, Gérard, Jensen, Perry, Sahbani, ...)

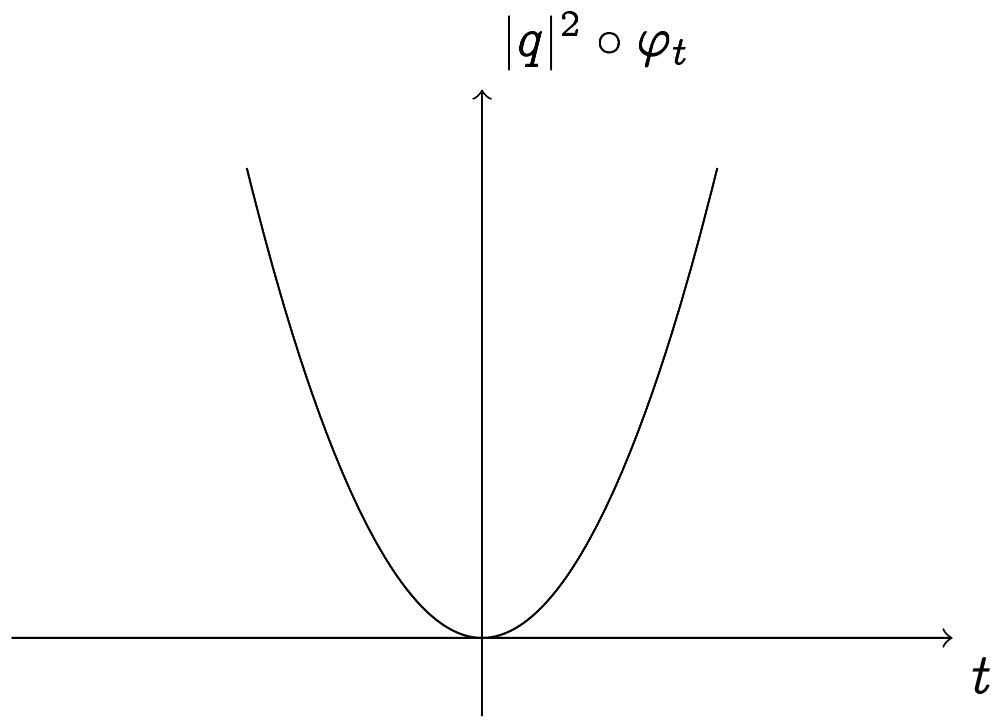
1.1 Classical mechanics as a motivation

- M, symplectic/Poisson manifold with Poisson bracket $\{\cdot, \cdot\}$
- $H \in C^\infty(M)$, Hamiltonian with complete flow $\{\varphi_t\}_{t\in\mathbb{R}}$
- Hamiltonian evolution equation for an observable $f \in C^\infty(M)$:

$$\frac{\mathrm{d}}{\mathrm{d}t} f \circ \varphi_t = \{f, H\} \circ \varphi_t, \qquad t \in \mathbb{R}.$$

For instance, if $H(q,p) := |p|^2 + V(q)$ on $M := T^*\mathbb{R}^d$ with $V \in C_c^\infty(\mathbb{R}^d)$, let's say that we don't want orbits bounded in $|q|^2$.

We want something like:



Since, $\frac{\mathrm{d}^2}{\mathrm{d}t^2}|q|^2 \circ \varphi_t = \big\{\{|q|^2, H\}, H\big\} \circ \varphi_t$, it is sufficient to check that

$$\big\{\{|q|^2, H\}, H\big\} \geq \delta > 0.$$

In the example $H(q,p) = |p|^2 + V(q)$, we get

$$\begin{aligned}\big\{\{|q|^2, H\}, H\big\} &= \big\{\{|q|^2, |p|^2 + V(q)\}, H\big\} \\ &= \big\{4(q \cdot p), |p|^2 + V(q)\big\} \\ &= 8\,|p|^2 - 4\,q \cdot (\nabla V)(q).\end{aligned}$$

Thus, $|p|^2 > \frac{1}{2} \sup_{q \in \mathbb{R}^n} \big|q \cdot (\nabla V)(q)\big|$ implies $\lim\limits_{|t| \to \infty} |q|^2 \circ \varphi_t = +\infty$.

(If the kinetic energy $|p|^2$ is large enough, all the trajectories go to infinity . . .)

To some extent, the idea behind commutators methods for self-adjoint operators is to translate the last example into the language of the (quantum) Hilbertian theory with the following heuristic dictionnary in mind:

Poisson manifold M	$\longleftrightarrow$	Hilbert space $\mathcal{H}$
Poisson bracket $\{\cdot,\cdot\}$	$\longleftrightarrow$	commutator $i[\cdot,\cdot]$
Hamiltonian $H \in C^\infty(M)$	$\longleftrightarrow$	self-adjoint operator H in $\mathcal{H}$
$\frac{\mathrm{d}}{\mathrm{d}t} f \circ \varphi_t = \{f, H\} \circ \varphi_t$	$\longleftrightarrow$	$\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{e}^{-itH} F \mathrm{e}^{itH} = \mathrm{e}^{itH}[iF, H]\mathrm{e}^{-itH}$
bounded orbits of H	$\longleftrightarrow$	eigenvalues of H

1.2 Self-adjoint operators

References:

- W. O. Amrein, Hilbert Space Methods In Quantum Mechanics, EPFL Press, 2009
- M. Reed and B. Simon, Methods of modern mathematical physics. volumes I-IV, Academic Press, 1980
- J. Weidmann, Linear Operators in Hilbert Spaces, Springer Verlag, 1980

An operator H with dense domain $\mathcal{D}(H)$ in a Hilbert space $\mathcal{H}$ is symmetric if

$$\langle H\varphi, \psi\rangle = \langle \varphi, H\psi\rangle \quad \text{for all } \varphi, \psi \in \mathcal{D}(H).$$

A vector $\eta \in \mathcal{H}$ belongs to $\mathcal{D}(H^*)$ if there exists $\eta^* \in \mathcal{H}$ such that

$$\langle \eta^*, \varphi\rangle = \langle \eta, A\varphi\rangle \quad \text{for all } \varphi \in \mathcal{D}(H).$$

In this case, one sets $H^*\eta := \eta^*$ and one calls H^* the adjoint of H.

A symmetric operator H is self-adjoint if

$$\{H, \mathcal{D}(H)\} = \{H^*, \mathcal{D}(H^*)\},$$

which is verified if and only if the ranges $\mathrm{Ran}(H \pm i) = \mathcal{H}$.

If H is self-adjoint, then the set $\mathcal{D}(H)$ equipped with the inner product

$$\langle \varphi, \psi \rangle_{\mathcal{D}(H)} := \langle \varphi, \psi \rangle + \langle H\varphi, H\psi \rangle, \quad \varphi, \psi \in \mathcal{D}(H),$$

and the induced norm

$$\|\varphi\|^2_{\mathcal{D}(H)} := \langle \varphi, \varphi \rangle_{\mathcal{D}(H)}, \quad \varphi \in \mathcal{D}(H),$$

defines a Hilbert space (a complete inner space).

A subspace $\mathscr{D} \subset \mathcal{D}(H)$ is a core for H if the closure of $\mathscr{D}$ in $\mathcal{D}(H)$ is equal to $\mathcal{D}(H)$; that is,

$$\overline{\mathscr{D}}^{\|\cdot\|_{\mathcal{D}(H)}} = \mathcal{D}(H).$$

Example 1.1. *The multiplication operator Q in $\mathcal{H} := \mathsf{L}^2(\mathbb{R})$ given by*

$$(Q\varphi)(x) := x\,\varphi(x), \quad \varphi \in \mathcal{H}_1(\mathbb{R}) := \left\{\varphi \in \mathcal{H} \mid \int_{\mathbb{R}} \left(1 + |x|^2\right)|\varphi(x)|^2 < \infty\right\},$$

is self-adjoint.

Example 1.2. *The operator P in $\mathcal{H} := \mathsf{L}^2(\mathbb{R})$ given by*

$$(P\varphi)(x) := -i\,\varphi'(x), \quad \varphi \in \mathcal{H}^1(\mathbb{R}) := \mathscr{F}\mathcal{H}_1(\mathbb{R}),$$

with $\mathscr{F}$ the 1-dimensional Fourier transform, is self-adjoint.

(the operator P is just the Fourier transform of the operator Q; that is, $Q = \mathscr{F} P \mathscr{F}^{-1}$)

The space $\mathscr{S}(\mathbb{R})$ of Schwartz functions on $\mathbb{R}$ is a core for Q and P, since $\mathscr{S}(\mathbb{R})$ is dense in the (Sobolev) spaces $\mathcal{H}_1(\mathbb{R})$ and $\mathcal{H}^1(\mathbb{R})$.

Let $\mathscr{B}(\mathcal{H})$ be the set of bounded operators on $\mathcal{H}$ and let H be a self-adjoint operator H in $\mathcal{H}$.

The set

$$\rho(H) := \{z \in \mathbb{C} \mid (H - z)^{-1} \text{ exists and belongs to } \mathscr{B}(\mathcal{H})\}$$

is the resolvent set of H; it is an open subset of $\mathbb{C}$.

The set $\sigma(H) := \mathbb{C} \setminus \rho(H)$ is the spectrum of H; it is a closed subset of $\mathbb{R}$.

A spectral family on a Hilbert space $\mathcal{H}$ is a function $E : \mathbb{R} \to \mathscr{B}(\mathcal{H})$ such that

- $E(\lambda)$ is an orthogonal projection for each $\lambda \in \mathbb{R}$, *i.e.*,

$$E(\lambda) = E(\lambda)^* = E(\lambda)^2 \quad \text{for each } \lambda \in \mathbb{R},$$

- $E(\mu) \le E(\lambda)$ for all $\mu \le \lambda$, *i.e.*,

$$\big\langle \varphi, E(\mu)\varphi \big\rangle \le \big\langle \varphi, E(\lambda)\varphi \big\rangle \quad \text{for all } \varphi \in \mathcal{H},\ \mu \le \lambda \quad \text{(monotonicity)},$$

- $\operatorname{s-lim}_{\varepsilon \searrow 0} E(\lambda + \varepsilon) = E(\lambda)$ for each $\lambda \in \mathbb{R}$ (right continuity),
- $\operatorname{s-lim}_{\lambda \to -\infty} E(\lambda) = 0$ and $\operatorname{s-lim}_{\lambda \to \infty} E(\lambda) = 1$.

For intervals, one defines the spectral measure

$$E\big((a,b]\big) := E(b) - E(a), \quad E\big((a,b)\big) := \operatorname{s-lim}_{\varepsilon \searrow 0} E(b-\varepsilon) - E(a), \quad \text{etc.}$$

and one extends these definitions to $E(\mathcal{V})$ for any Borel set $\mathcal{V} \subset \mathbb{R}$.

Theorem 1.3 (Spectral theorem). *A self-adjoint operator H in a Hilbert space $\mathcal{H}$ admits exactly one spectral family E^H such that*

$$H = \int_{\mathbb{R}} \lambda\, E^H(\mathrm{d}\lambda),$$

with the strong integral $\int_{\mathbb{R}} \lambda\, \mathrm{d}E^H(\mathrm{d}\lambda)$ satisfying

$$\left\langle \varphi, \int_{\mathbb{R}} \lambda\, E^H(\mathrm{d}\lambda)\,\psi \right\rangle := \int_{\mathbb{R}} \lambda\, \langle \varphi, E^H(\mathrm{d}\lambda)\,\psi \rangle, \quad \varphi \in \mathcal{H},\ \psi \in \mathcal{D}(H).$$

Furthermore, one has for $-\infty < a < b < \infty$ that

$$E^H\big((a,b]\big) = \frac{1}{\pi}\, \text{s-}\lim_{\delta \searrow 0} \text{s-}\lim_{\varepsilon \searrow 0} \int_{a+\delta}^{b+\delta} \mathrm{d}\lambda\ \mathrm{Im}(H - \lambda - i\varepsilon)^{-1}.$$

(Stone's Formula)

Two comments:

- The support of the spectral family E^H is the set of points of non-constancy and coincides with the spectrum of H

  $$\operatorname{supp}\big(E^H\big) = \big\{\lambda \in \mathbb{R} \mid E^H(\lambda+\varepsilon) - E^H(\lambda-\varepsilon) \neq 0 \;\; \forall \varepsilon > 0\big\} = \sigma(H).$$

- Formally, one has

  $$\begin{aligned} \|H\psi\|^2 = \langle H\psi, H\psi\rangle &= \int_{\mathbb{R}} \lambda \int_{\mathbb{R}} \mu \,\big\langle E^H(\mathrm{d}\mu)\,\psi, E^H(\mathrm{d}\lambda)\,\psi\big\rangle \\ &= \int_{\mathbb{R}} \lambda \int_{\mathbb{R}} \mu \,\big\langle \psi, E^H(\mathrm{d}\mu \cap \mathrm{d}\lambda)\,\psi\big\rangle \\ &= \int_{\mathbb{R}} \lambda^2 \,\big\langle \psi, E^H(\mathrm{d}\lambda)\,\psi\big\rangle, \end{aligned}$$

  so that $\psi \in \mathcal{D}(H)$ if and only if $\int_{\mathbb{R}} \lambda^2 \big\langle \psi, E^H(\mathrm{d}\lambda)\,\psi\big\rangle < \infty$.

Example 1.4. *The spectral projection* $E^Q(\lambda)$ *of the operator* Q *in* $\mathcal{H} := \mathsf{L}^2(\mathbb{R})$ *is the operator of multiplication by the characteristic function* $\chi_{(-\infty,\lambda]}$, i.e.,

$$E^Q(\lambda)\varphi := \chi_{(-\infty,\lambda]}\varphi, \quad \varphi \in \mathcal{H}.$$

One verifies that

$$\sigma(Q) = \operatorname{supp}\big(E^Q\big) = \mathbb{R}.$$

Example 1.5. *The multiplication operator* $Q^2 := \sum_{j=1}^{d} Q_j^2$ *in* $\mathcal{H} := \mathsf{L}^2(\mathbb{R}^d)$ *given by*

$$\big(Q^2\varphi\big)(x) := x^2\,\varphi(x), \quad \varphi \in \mathcal{H}_2(\mathbb{R}^d),\; x^2 := \sum_{j=1}^{d} x_j^2,$$

is self-adjoint, and its spectral family is given by

$$E^{Q^2}(\lambda)\,\varphi := \begin{cases} \chi_{[-\lambda^{1/2},\lambda^{1/2}]}\,\varphi & \text{if } \lambda > 0 \\ 0 & \text{if } \lambda \le 0, \end{cases} \qquad \varphi \in \mathcal{H}.$$

One verifies that

$$\sigma\big(Q^2\big) = \operatorname{supp}\big(E^{Q^2}\big) = [0,\infty).$$

The Laplacian $-\triangle$ in $\mathcal{H} := \mathsf{L}^2(\mathbb{R}^d)$ satisfies on $\mathscr{S}(\mathbb{R}^d)$ (and thus on $\mathcal{H}^2(\mathbb{R}^d)$)

$$-\triangle = \sum_{j=1}^{d} P_j^2 \equiv P^2 = \mathscr{F}^{-1} Q^2 \mathscr{F},$$

with $\mathscr{F}$ the d-dimensional Fourier transform. So, one has

$$E^{-\triangle} = E^{\mathscr{F}^{-1} Q^2 \mathscr{F}} \overset{\text{(Stone)}}{=} \mathscr{F}^{-1} E^{Q^2} \mathscr{F}.$$

Let $\mathcal{A}_{\rm B}$ be the Borel σ-algebra of $\mathbb{R}$ and $|\mathcal{V}|$ be the Lebesgue measure of $\mathcal{V} \in \mathcal{A}_{\rm B}$.

If H is a self-adjoint operator in $\mathcal{H}$, one has the orthogonal decompositions

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_{\rm p}(H) \oplus \mathcal{H}_{\rm sc}(H) \oplus \mathcal{H}_{\rm ac}(H) \\ H &= H|_{\mathcal{H}_{\rm p}(H)} \oplus H|_{\mathcal{H}_{\rm sc}(H)} \oplus H|_{\mathcal{H}_{\rm ac}(H)}, \end{aligned}$$

with

$$\begin{aligned} \mathcal{H}_{\rm p}(H) &:= \overline{\rm Span}\big\{\text{eigenvectors of } H\big\} \\ \mathcal{H}_{\rm sc}(H) &:= \big\{\varphi \in \mathcal{H} \mid \lambda \mapsto \|E^H(\lambda)\varphi\|^2 \text{ is continuous} \\ &\qquad \text{and } \exists \mathcal{V} \in \mathcal{A}_{\rm B} \text{ with } |\mathcal{V}| = 0 \text{ and } E^H(\mathcal{V})\varphi = \varphi\big\} \\ \mathcal{H}_{\rm ac}(H) &:= \big\{\varphi \in \mathcal{H} \mid \lambda \mapsto \|E^H(\lambda)\varphi\|^2 \text{ is absolutely continuous}\big\}. \end{aligned}$$

The subspaces $\mathcal{H}_{\mathrm{p}}(H)$, $\mathcal{H}_{\mathrm{sc}}(H)$, $\mathcal{H}_{\mathrm{ac}}(H)$ are the pure point subspace of H, the singular continuous subspace of H and the absolutely continuous subspace of H.

The decomposition of $\mathcal{H}$ induces a decomposition of $\sigma(H)$

$$\sigma(H) = \sigma_{\mathrm{p}}(H) \cup \sigma_{\mathrm{sc}}(H) \cup \sigma_{\mathrm{ac}}(H),$$

with

$$\begin{aligned} \sigma_{\mathrm{p}}(H) &:= \sigma\big(H|_{\mathcal{H}_{\mathrm{p}}(H)}\big) \text{ the pure point spectrum of } H, \\ \sigma_{\mathrm{sc}}(H) &:= \sigma\big(H|_{\mathcal{H}_{\mathrm{sc}}(H)}\big) \text{ the singular continuous spectrum of } H, \\ \sigma_{\mathrm{ac}}(H) &:= \sigma\big(H|_{\mathcal{H}_{\mathrm{ac}}(H)}\big) \text{ the absolutely continuous spectrum of } H. \end{aligned}$$

The sets $\sigma_{\mathrm{p}}(H)$, $\sigma_{\mathrm{sc}}(H)$, $\sigma_{\mathrm{ac}}(H)$ are closed and (in general) not mutually disjoint.

Example 1.6. *For each* $\lambda \in \mathbb{R}$ *and* $\varphi \in \mathcal{H} := \mathsf{L}^2(\mathbb{R})$, *one has*

$$\left\| E^Q(\lambda)\varphi \right\|^2 = \left\| \chi_{(-\infty,\lambda]}\varphi \right\|^2 = \int_{-\infty}^{\lambda} \mathrm{d}x\, |\varphi(x)|^2 = \text{integral of a } \mathsf{L}^1\text{-function} = \text{absolutely continuous function.}$$

So, $\mathcal{H} = \mathcal{H}_{\mathrm{ac}}(Q)$ *and* Q *has purely absolutely continuous spectrum* $\sigma(Q) = \sigma_{\mathrm{ac}}(Q) = \mathbb{R}$.

In fact, Q *has Lebesgue spectrum since*

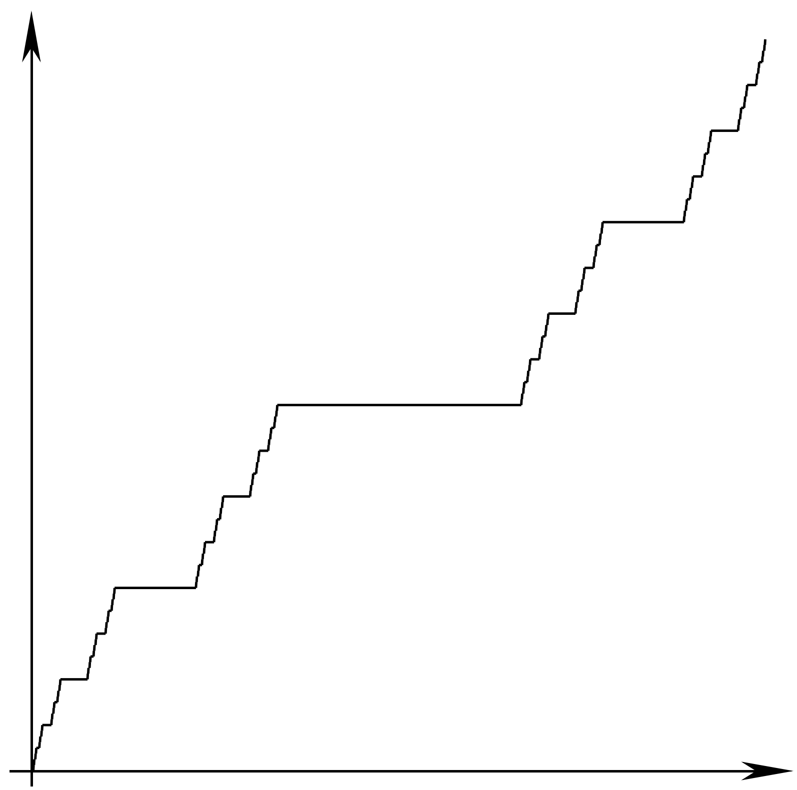
$$\mathrm{e}^{itP}\,\mathrm{e}^{isQ}\,\mathrm{e}^{-itP} = \mathrm{e}^{ist}\,\mathrm{e}^{isQ},\quad s,t \in \mathbb{R} \iff \mathrm{e}^{itP}\,Q\,\mathrm{e}^{-itP} = Q+t,\quad t \in \mathbb{R}.$$

(. . . Stone-von Neumann theorem . . .)

Example 1.7. *Let* $f : [0,1] \to [0,1]$ *be the Cantor function, and let*

$$M_f \varphi := f\varphi, \quad \varphi \in \mathcal{H} := \mathsf{L}^2([0,1]),$$

be the corresponding bounded multiplication operator.



The spectral family of M_f *is*

$$E^{M_f}(\lambda)\varphi := \begin{cases} \chi_{f^{-1}([0,\lambda])}\varphi & \text{if } \lambda \in [0,1] \\ 0 & \text{if } \lambda \in \mathbb{R} \setminus [0,1], \end{cases} \qquad \varphi \in \mathcal{H}.$$

One verifies that

$$\sigma(M_f) = \operatorname{supp}\big(E^{Q^2}\big) = \text{Cantor ternary set}$$

and that the function

$$[0,1] \ni \lambda \mapsto \left\|E^{M_f}(\lambda)\varphi\right\|^2 = \left\|\chi_{f^{-1}([0,\lambda])}\varphi\right\|^2 = \int_0^1 \mathrm{d}x\, \chi_{f^{-1}([0,\lambda])}(x)|\varphi(x)|^2$$

is continuous but not absolutely continuous.

So, $\mathcal{H} = \mathcal{H}_{\mathrm{sc}}(M_f)$ *and* M_f *has purely singular continuous spectrum* $\sigma(M_f) = \sigma_{\mathrm{sc}}(M_f) = \text{Cantor ternary set}$.

An interesting link between spectral theory and dynamics is provided by the following:

Theorem 1.8 (RAGE theorem). *Let H be a self-adjoint operator in a Hilbert space $\mathcal{H}$ and let $C \in \mathscr{B}(\mathcal{H})$ be such that $C(H+i)^{-1}$ is compact. Then,*

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathrm{d}t \left\| C\, \mathrm{e}^{-itH} \varphi \right\|^2 = 0 \qquad \textit{for all } \varphi \in \mathcal{H}_{\mathrm{sc}}(H) \oplus \mathcal{H}_{\mathrm{ac}}(H).$$

RAGE theorem says that, as time evolves, the state φ in the continuous subspace of H escapes (in Cesàro mean) from the range of the operator C.

(the typical example is when H is a Schrödinger operator in $\mathbb{R}^d$ and C the orthogonal projection onto a compact subset of $\mathbb{R}^d$)

1.3 Commutator methods for self-adjoint operators

References:

- W. O. Amrein, A. Boutet de Monvel and V. Georgescu, C_0-groups, commutator methods and spectral theory of N-body Hamiltonians, Birhäuser, 1996
- É. Mourre, Absence of singular continuous spectrum for certain selfadjoint operators, Comm. Math. Phys., 1980/81.
- J. Sahbani, The conjugate operator method for locally regular Hamiltonians, J. Operator Theory, 1997.

- $\mathcal{H}$, Hilbert space with norm $\|\cdot\|$ and scalar product $\langle\cdot,\cdot\rangle$
- $\mathscr{B}(\mathcal{H})$, set of bounded linear operators on $\mathcal{H}$
- $\mathscr{K}(\mathcal{H})$, set of compact operators on $\mathcal{H}$
- A, H, self-adjoint operators in $\mathcal{H}$ with domains $\mathcal{D}(A), \mathcal{D}(H)$, spectral families $E^A(\cdot)$, $E^H(\cdot)$ and spectra $\sigma(A), \sigma(H)$
- The adjoint space of a Banach space $\mathcal{B}$ is defined by

$$\mathcal{B}^* := \big\{\text{anti-linear continuous functions } \phi : \mathcal{B} \to \mathbb{C}\big\}$$

$$\|\phi\|_{\mathcal{B}^*} := \sup\big\{|\phi(\varphi)| \mid \varphi \in \mathcal{B},\ \|\varphi\|_{\mathcal{B}} \leq 1\big\}$$

Definition 1.9. *An operator $S \in \mathscr{B}(\mathcal{H})$ satisfies $S \in C^k(A)$ if the map*

$$\mathbb{R} \ni t \mapsto \mathrm{e}^{-itA}\, S\, \mathrm{e}^{itA} \in \mathscr{B}(\mathcal{H})$$

is strongly of class C^k.

In other terms, $S \in C^k(A)$ if there exist

$$B_0(t) \equiv \mathrm{e}^{-itA}\, S\, \mathrm{e}^{itA}, B_1(t), B_2(t), \ldots, B_k(t) \in \mathscr{B}(\mathcal{H}), \quad t \in \mathbb{R},$$

such that

$$\lim_{h \to 0} \left\| \frac{B_j(t+h) - B_j(t)}{h}\, \varphi - B_{j+1}(t)\, \varphi \right\| = 0 \quad \text{for all } t \in \mathbb{R},\ \varphi \in \mathcal{H},$$

for $j = 0, 1, \ldots, k-1$.

$S \in C^1(A)$ if and only if the quadratic form

$$\mathcal{D}(A) \ni \varphi \mapsto \langle A\varphi, S\varphi \rangle - \langle \varphi, SA\varphi \rangle \in \mathbb{C}$$

is continuous for the topology induced by $\mathcal{H}$ on $\mathcal{D}(A)$; that is, if

$$\big| \langle A\varphi, S\varphi \rangle - \langle \varphi, SA\varphi \rangle \big| \leq \text{Const.}\, \|\varphi\|^2 \quad \text{for all } \varphi \in \mathcal{D}(A).$$

The bounded operator corresponding to the continuous extension of the quadratic form is denoted by $[A, S]$, and one has

$$-[iA, S] = \text{s-}\frac{\mathrm{d}}{\mathrm{d}t}\, \mathrm{e}^{-itA}\, S\, \mathrm{e}^{itA} \Big|_{t=0} \in \mathscr{B}(\mathcal{H}).$$

Example 1.10. *Let* $f \in \mathsf{L}^\infty(\mathbb{R})$ *be an absolutely continuous function with* $f' \in \mathsf{L}^\infty(\mathbb{R})$*, and let*

$$M_f \varphi := f\varphi, \quad \varphi \in \mathcal{H} := \mathsf{L}^2(\mathbb{R}),$$

be the corresponding bounded multiplication operator.

Then, one has for each $\varphi \in \mathcal{H}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{e}^{-itP}\,M_f\,\mathrm{e}^{itP}\,\varphi = \frac{\mathrm{d}}{\mathrm{d}t}\,M_{f(\,\cdot\,-t)}\,\varphi = -M_{f'(\,\cdot\,-t)}\,\varphi,$$

and thus $M_f \in C^1(P)$ *with* $[iP, M_f] = M_{f'}$*.*

In the case of (unbounded) self-adjoint operators, we have a similar definition:

Definition 1.11. *A self-adjoint operator H is of class $C^k(A)$ if $(H-z)^{-1} \in C^k(A)$ for some $z \in \rho(H)$.*

If H is of class $C^1(A)$, then

$$\begin{aligned} \big[A,(H-z)^{-1}\big] &= (H-z)^{-1}[H-z,A](H-z)^{-1} \\ &= (H-z)^{-1}[H,A](H-z)^{-1}, \end{aligned}$$

with $[H,A]$ the bounded operator from $\mathcal{D}(H)$ to $\mathcal{D}(H)^*$ associated with the continuous extension to $\mathcal{D}(H)$ of the quadratic form

$$\mathcal{D}(H) \cap \mathcal{D}(A) \ni \varphi \mapsto \big\langle H\varphi, A\varphi\big\rangle - \big\langle A\varphi, H\varphi\big\rangle \in \mathbb{C}.$$

Theorem 1.12 (Virial Theorem)**.** *Let A, H be self-adjoint operators with H of class $C^1(A)$. Then,*

$$E^H(\{\lambda\})[A,H]E^H(\{\lambda\}) = 0 \quad \textit{for each } \lambda \in \mathbb{R}.$$

Thus, one has $\langle \varphi, [A,H]\varphi \rangle = 0$ if φ is an eigenvector of H.

Proof. We must show that if $\varphi_1, \varphi_2 \in \mathcal{D}(H)$ satisfy $H\varphi_j = \lambda\varphi_j$ for some $\lambda \in \mathbb{R}$, then $\langle \varphi_1, [A, H]\varphi_2 \rangle = 0$.

But,

$$\begin{aligned}
&\langle \varphi_1, [A, H]\varphi_2 \rangle \\
&= \langle (\lambda - i)(H - i)^{-1}\varphi_1, [A, H](\lambda + i)(H + i)^{-1}\varphi_2 \rangle \\
&= -(\lambda + i)^2 \langle \varphi_1, \left[A, (H + i)^{-1}\right]\varphi_2 \rangle \\
&= -(\lambda + i)^2 \lim_{\tau \to 0} \langle \varphi_1, \left[\tfrac{1}{i\tau}(\mathrm{e}^{i\tau A} - 1), (H + i)^{-1}\right]\varphi_2 \rangle \\
&= -(\lambda + i)^2 \lim_{\tau \to 0} \frac{1}{i\tau}\left\{ \langle \varphi_1, \mathrm{e}^{i\tau A}(H + i)^{-1}\varphi_2 \rangle - \langle (H - i)^{-1}\varphi_1, \mathrm{e}^{i\tau A}\varphi_2 \rangle \right\} \\
&= -(\lambda + i)^2 \lim_{\tau \to 0} \frac{1}{i\tau}\{0\}.
\end{aligned}$$

□

Corollary 1.13 (Point spectrum of H). *Let A, H be self-adjoint operators with H of class $C^1(A)$. Assume there exist a Borel set $I \subset \mathbb{R}$, a number $a > 0$ and $K \in \mathscr{K}(\mathcal{H})$ such that*

$$E^H(I)[iH, A]E^H(I) \geq a\, E^H(I) + K. \tag{1.1}$$

Then, H has at most finitely many eigenvalues in I (multiplicities counted).

Some comments:

- If I is bounded, one has

$$\underbrace{E^H(I)}_{\in \mathscr{B}(\mathcal{D}(H)^*,\mathcal{H})} \quad \underbrace{[iH, A]}_{\in \mathscr{B}(\mathcal{D}(H),\mathcal{D}(H)^*)} \quad \underbrace{E^H(I)}_{\in \mathscr{B}(\mathcal{H},\mathcal{D}(H))} \quad \in \mathscr{B}(\mathcal{H}).$$

- If I is not bounded, the inequality (1.1) holds in the sense of quadratic forms on $\mathcal{D}(H)$.

- The inequality (1.1) is called a Mourre estimate.

Proof. If $\varphi \in \mathcal{H}$ is an eigenvector of H with $\|\varphi\| = 1$ and with eigenvalue in I, the Mourre inequality (1.1) implies that

$$0 \geq a\langle \varphi, E^H(I)\varphi \rangle + \langle \varphi, K\varphi \rangle \implies \langle \varphi, K\varphi \rangle \leq -a.$$

Now, if the claim were false, there would exist an infinite orthonormal sequence $\{\varphi_j\}$ of eigenvectors of H in $E^H(I)\mathcal{H}$. In particular, one would have $\text{w-}\lim_{j\to\infty} \varphi_j = 0$. Since $K \in \mathscr{K}(\mathcal{H})$, this would imply that $\lim_{j\to\infty} \langle \varphi_j, K\varphi_j \rangle = 0$, which contradicts the inequality $\langle \varphi_j, K\varphi_j \rangle \leq -a < 0$. □

Note that the proof shows that if $K = 0$, then H is purely continuous in $I \cap \sigma(H)$.

Example 1.14 (Finite dimension). *If* $\dim(\mathcal{H}) < \infty$*, then* A, H *are hermitian matrices and* $H \in C^\infty(A)$*.*

Furthermore, one has

$$[iH, A] = 1 + \big([iH, A] - 1\big) = 1 + \text{compact operator},$$

and the corollary implies (without surprise) that H *has at most finitely many eigenvalues in* $\sigma(H)$*.*

Definition 1.15. *$S \in C^{1+0}(A)$ if $S \in C^1(A)$ and*

$$\int_0^1 \frac{\mathrm{d}t}{t} \left\| \mathrm{e}^{-itA}[A,S]\,\mathrm{e}^{itA} - [A,S] \right\|_{\mathscr{B}(\mathcal{H})} < \infty.$$

Similarly, a self-adjoint operator H is of class $C^{1+0}(A)$ if $(H-z)^{-1} \in C^{1+0}(A)$ for some $z \in \rho(H)$.

If we regard $C^1(A)$, $C^{1+0}(A)$ and $C^2(A)$ as subspaces of $\mathscr{B}(\mathcal{H})$, we have the inclusions

$$C^2(A) \subset C^{1+0}(A) \subset C^1(A) \subset \mathscr{B}(\mathcal{H}).$$

Example 1.16. *Let* $f \in \mathsf{L}^\infty(\mathbb{R})$ *be an absolutely continuous function with* $f' \in \mathsf{L}^\infty(\mathbb{R})$ *Dini-continuous, and let* M_f *be the corresponding multiplication operator in* $\mathcal{H} := \mathsf{L}^2(\mathbb{R})$.

Then, we know that $M_f \in C^1(P)$ *with* $[iP, M_f] = M_{f'}$*, and*

$$\begin{aligned}\int_0^1 \frac{\mathrm{d}t}{t} \big\| \mathrm{e}^{-itP}[P, M_f]\,\mathrm{e}^{itP} - [P, M_f] \big\|_{\mathscr{B}(\mathcal{H})} &= \int_0^1 \frac{\mathrm{d}t}{t} \big\| M_{f'(\,\cdot\,-t)-f'} \big\|_{\mathscr{B}(\mathcal{H})} \\ &= \int_0^1 \frac{\mathrm{d}t}{t} \big\| f'(\,\cdot\,-t) - f' \big\|_{\mathsf{L}^\infty(\mathbb{R})} \\ &< \infty\end{aligned}$$

due to the Dini-continuity of f'. *So, one has* $M_f \in C^{1+0}(P)$.

Spectral result of Mourre (and Amrein, Boutet de Monvel, Georgescu, Sahbani, . . .)

Theorem 1.17 (Spectral properties of H). *Let H be of class $C^{1+0}(A)$. Assume there exist an open set $I \subset \mathbb{R}$, a number $a > 0$ and $K \in \mathscr{K}(\mathcal{H})$ such that*

$$E^H(I)[iH, A]E^H(I) \geq a\,E^H(I) + K.$$

Then, H has at most finitely many eigenvalues in I (multiplicities counted), and H has no singular continuous spectrum in I.

Some comments:

- the operator A is called a conjugate operator for H on I
- if $K = 0$, then H is purely absolutely continuous in $I \cap \sigma(H)$
- if H has a spectral gap or satisfies an additional invariance assumption, then one can replace the condition $C^{1+0}(A)$ by a weaker condition $C^{1,1}(A)$

Sketch of the proof of Mourre (i)

One has for $\mu \in \sigma(H)$ and $\varepsilon \in \mathbb{R}$ that

$$\left\|(H-\mu-i\varepsilon)^{-1}\right\| = \left\|x \mapsto (x-\mu-i\varepsilon)^{-1}\right\|_{\mathsf{L}^\infty(\mathbb{R})} = |\varepsilon|^{-1}.$$

Thus, $(H-\mu-i\varepsilon)^{-1}$ cannot have a limit in $\mathscr{B}(\mathcal{H})$ as $\varepsilon \to \pm 0$.

However, for some $\varphi \in \mathcal{H} \setminus \{0\}$, the holomorphic function

$$F : \rho(H) \to \mathbb{C}, \quad z \mapsto \left\langle \varphi, (H-z)^{-1}\varphi \right\rangle,$$

may have a limit

$$F(\mu) := \lim_{\varepsilon \searrow 0} F(\mu + i\varepsilon)$$

uniformly on each interval $[a,b] \subset I$.

In such a case, Stone's Formula and Lebesgue's dominated convergence theorem imply for $\lambda \in (a,b]$ that

$$\left\|E^H\big((a,\lambda]\big)\varphi\right\|^2 = \left\langle \varphi, E^H\big((a,\lambda]\big)\varphi\right\rangle = \frac{1}{\pi}\int_a^\lambda \mathrm{d}\mu \ \mathrm{Im}\, F(\mu).$$

But, F is continuous on $[a,b]$ due to the uniform convergence of the sequence $F_\varepsilon(\,\cdot\,) := \left\langle \varphi, \big(H - (\,\cdot\,) - i\varepsilon\big)^{-1}\varphi\right\rangle$. Thus,

$$\mathrm{Im}\, F(\mu) \in \mathsf{L}^1([a,b]) \quad \text{and} \quad E^H(I)\varphi \in \mathcal{H}_{\mathrm{ac}}(H).$$

Therefore, if there is a dense set of vectors $\varphi \in \mathcal{H}$ satisfying what precedes, then $E^H(I)\mathcal{H} \subset \mathcal{H}_{\mathrm{ac}}(H)$ and H is purely absolutely continuous in $I \cap \sigma(H)$.

Sketch of the proof of Mourre (ii)

Let's show the existence of the limit $\lim_{\varepsilon \searrow 0} F(\mu + i\varepsilon)$ in the homogeneous case $[iH, A] = H$.

(in such case, one has $\mathrm{e}^{-itA}\, H\, \mathrm{e}^{itA} = \mathrm{e}^{t}\, H$, and thus we already know that H has homogeneous spectrum on $\mathbb{R} \setminus \{0\}$)

One has for $z \in \rho(H)$

$$
\begin{aligned}
z \frac{\mathrm{d}}{\mathrm{d}z} (H - z)^{-1} = z\,(H - z)^{-2} &= (H - z)^{-1} H (H - z)^{-1} - (H - z)^{-1} \\
&= \big[iA, (H - z)^{-1}\big] - (H - z)^{-1}
\end{aligned}
$$

which gives for $\varphi \in \mathcal{D}(A)$

$$
z \frac{\mathrm{d}}{\mathrm{d}z} F(z) = -F(z) - \big\langle iA\varphi, (H - z)^{-1} \big\rangle - \big\langle (H - \bar{z})^{-1} \varphi, iA\varphi \big\rangle.
$$

But, if $z = \mu + i\varepsilon$ with $\varepsilon > 0$, then

$$\begin{aligned} \left\|(H-\mu-i\varepsilon)^{-1}\varphi\right\|^2 &= \left\|(H-\mu+i\varepsilon)^{-1}\varphi\right\|^2 \\ &= \left\langle\varphi, |H-\mu-i\varepsilon|^{-2}\varphi\right\rangle \\ &= \left|\left\langle\varphi, \varepsilon^{-1}\,\mathsf{Im}(H-\mu-i\varepsilon)^{-1}\varphi\right\rangle\right| \\ &= \varepsilon^{-1}\left|\,\mathsf{Im}\,F(\mu+i\varepsilon)\right|. \end{aligned}$$

Thus, we get for $z = \mu + i\varepsilon$ with $\mu \neq 0$ fixed and $\varepsilon > 0$ that

$$
\begin{aligned}
&\left| z \frac{\mathrm{d}}{\mathrm{d}z} F(z) \right| = \left| -F(z) - \left\langle iA\varphi, (H-z)^{-1} \right\rangle - \left\langle (H-\bar{z})^{-1}\varphi, iA\varphi \right\rangle \right| \\
&\Longrightarrow |\mu + i\varepsilon| \left| \frac{\mathrm{d}}{\mathrm{d}\varepsilon} F(\mu+i\varepsilon) \right| \leq \left| F(\mu+i\varepsilon) \right| + 2 \|A\varphi\| \left\| (H-\mu-i\varepsilon)^{-1}\varphi \right\| \\
&\Longrightarrow \left| \frac{\mathrm{d}}{\mathrm{d}\varepsilon} F(\mu+i\varepsilon) \right| \leq \frac{1}{|\mu|} \big(\|\varphi\| + 2\|A\varphi\| \big) \left\| (H-\mu-i\varepsilon)^{-1}\varphi \right\| \\
&\Longrightarrow \left| \frac{\mathrm{d}}{\mathrm{d}\varepsilon} F(\mu+i\varepsilon) \right| \leq \frac{1}{|\mu|} \big(\|\varphi\| + 2\|A\varphi\| \big) \varepsilon^{-1/2} \left| F(\mu+i\varepsilon) \right|^{1/2}.
\end{aligned}
$$

Now,

$$\big|F(\mu+i\varepsilon)\big| \geq \big|\operatorname{Im} F(\mu+i\varepsilon)\big| = \varepsilon\left\|(H-\mu-i\varepsilon)^{-1}\varphi\right\|^2 > 0$$

if $\varepsilon > 0$ and $\varphi \neq 0$.

So, one can divide the last inequality by $\big|F(\mu+i\varepsilon)\big|^{1/2}$ to get

$$\frac{\big|\frac{\mathrm{d}}{\mathrm{d}\varepsilon} F(\mu+i\varepsilon)\big|}{\big|F(\mu+i\varepsilon)\big|^{1/2}} \leq \frac{1}{|\mu|}\big(\|\varphi\| + 2\|A\varphi\|\big)\,\varepsilon^{-1/2}$$

$$\iff \left|\frac{\mathrm{d}}{\mathrm{d}\varepsilon} F(\mu+i\varepsilon)^{1/2}\right| \leq \frac{1}{|\mu|}\big(\|\varphi\| + 2\|A\varphi\|\big)\,\frac{1}{2\varepsilon^{1/2}}$$

$$\overset{\int_\varepsilon^1 \mathrm{d}\varepsilon}{\Longrightarrow} \left|F(\mu+i)^{1/2} - F(\mu+i\varepsilon)^{1/2}\right| \leq \frac{1}{|\mu|}\big(\|\varphi\| + 2\|A\varphi\|\big)\,(1-\varepsilon^{1/2})$$

$$\overset{\varepsilon\in(0,1)}{\Longrightarrow} \big|F(\mu+i\varepsilon)\big|^{1/2} \leq \big|F(\mu+i)\big|^{1/2} + \frac{1}{|\mu|}\big(\|\varphi\| + 2\|A\varphi\|\big).$$

Putting the last estimate in the inequality

$$\left|\frac{\mathrm{d}}{\mathrm{d}\varepsilon}F(\mu+i\varepsilon)\right| \le \frac{1}{|\mu|}\big(\|\varphi\|+2\|A\varphi\|\big)\,\varepsilon^{-1/2}\big|F(\mu+i\varepsilon)\big|^{1/2},$$

one gets for each $|\mu| \ge \delta > 0$ and $\varepsilon \in (0,1)$

$$\begin{aligned}
&\left|\frac{\mathrm{d}}{\mathrm{d}\varepsilon}F(\mu+i\varepsilon)\right| \\
&\le \frac{1}{|\mu|}\big(\|\varphi\|+2\|A\varphi\|\big)\,\varepsilon^{-1/2}\left\{\big|F(\mu+i)\big|^{1/2}+\frac{1}{|\mu|}\big(\|\varphi\|+2\|A\varphi\|\big)\right\} \\
&\le \frac{1}{\delta}\big(\|\varphi\|+2\|A\varphi\|\big)\,\varepsilon^{-1/2}\left\{\|\varphi\|+\frac{1}{\delta}\big(\|\varphi\|+2\|A\varphi\|\big)\right\} \\
&\le \mathrm{c}(\delta,\varphi)\,\varepsilon^{-1/2}\big(\|\varphi\|^2+\|A\varphi\|^2\big).
\end{aligned}$$

It follows that $\{F(\mu + i/m)\}_{m\in\mathbb{N}^*}$ is a Cauchy sequence since

$$\begin{aligned}
\big|F(\mu + i/m) - F(\mu + i/n)\big| &= \left|\int_{1/n}^{1/m} \mathrm{d}\varepsilon \, \frac{\mathrm{d}}{\mathrm{d}\varepsilon} F(\mu + i\varepsilon)\right| \\
&\leq \mathrm{c}(\delta, \varphi)\big(\|\varphi\|^2 + \|A\varphi\|^2\big) \left|\int_{1/n}^{1/m} \mathrm{d}\varepsilon \, \varepsilon^{-1/2}\right| \\
&= 2\,\mathrm{c}(\delta, \varphi)\big(\|\varphi\|^2 + \|A\varphi\|^2\big)\big|m^{-1/2} - n^{-1/2}\big| \\
&\to 0 \quad \text{as } m, n \to \infty.
\end{aligned}$$

Thus, the limit $\lim_{\varepsilon \searrow 0} F(\mu + i\varepsilon)$ exists uniformly on $|\mu| \geq \delta$.

1.4 Schrödinger operators

Let M_V be the self-adjoint multiplication operator in $\mathcal{H} := \mathsf{L}^2(\mathbb{R})$ given by $V \in \mathsf{L}^\infty(\mathbb{R};\mathbb{R})$. Then, the 1-dimensional Schrödinger operator

$$H\varphi := -\triangle\varphi + M_V\varphi, \quad \varphi \in \mathcal{D}(H) := \mathcal{H}^2(\mathbb{R}),$$

is self-adjoint due to the Kato-Rellich theorem.

(self-adjointness is preserved under the perturbation by a bounded self-adjoint operator)

In quantum mechanics, the operator H describes a non-relativistic particle in $\mathbb{R}$ in presence of a scalar (electric) potential V.

Can we (under some assumptions on V) determine the spectral nature of H ?

Can we do it using commutator methods ?

The family of operators $\{U_t\}_{t\in\mathbb{R}}$ in $\mathcal{H}$ given by

$$(U_t\,\varphi)(x) := \mathrm{e}^{t/2}\,\varphi(\mathrm{e}^t\,x), \quad \varphi \in \mathscr{S}(\mathbb{R}),\ x, t \in \mathbb{R},$$

defines a strongly continuous unitary group (the dilation group).

The self-adjoint generator $A := i\big(\text{s-}\frac{\mathrm{d}}{\mathrm{d}t}\,U_t\big|_{t=0}\big)$ of $\{U_t\}_{t\in\mathbb{R}}$ acts as

$$A\varphi := \frac{1}{2}\big(QP + PQ\big)\varphi, \quad \varphi \in \mathscr{S}(\mathbb{R}).$$

The operator A is the quantum analogue of the classical observable $q \cdot p$ on $M := T^*\mathbb{R}$ which appeared at the beginning:

$$\begin{aligned} \{\{q^2, p^2 + V(q)\}, p^2 + V(q)\} &= \{4(q \cdot p), p^2 + V(q)\} \\ &= 8p^2 - 4q \cdot (\nabla V)(q). \end{aligned}$$

... just replace the observables q and p on $M := T^*\mathbb{R}$ by the self-adjoint operators Q and P in $\mathcal{H}$, and be cautious with the domains and the self-adjointness of the unbounded operators ...

One has

$$\begin{aligned}
&\mathrm{e}^{-itA}(-\triangle+i)^{-1}\,\mathrm{e}^{itA}\\
&=\mathscr{F}^{-1}(\mathscr{F}\,\mathrm{e}^{-itA}\,\mathscr{F}^{-1})\mathscr{F}(-\triangle+i)^{-1}\mathscr{F}^{-1}(\mathscr{F}\,\mathrm{e}^{itA}\,\mathscr{F}^{-1})\mathscr{F}\\
&=\mathscr{F}^{-1}U_{-t}\left(Q^2+i\right)^{-1}U_t\,\mathscr{F}\\
&=\mathscr{F}^{-1}\left((\mathrm{e}^{-t}\,Q)^2+i\right)^{-1}\mathscr{F}\\
&=\left(\mathrm{e}^{-2t}(-\triangle)+i\right)^{-1}.
\end{aligned}$$

Thus,

$$\text{s-}\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{e}^{-itA}(-\triangle+i)^{-1}\,\mathrm{e}^{itA}\Big|_{t=0}=(-\triangle+i)^{-1}\,2(-\triangle)(-\triangle+i)^{-1},$$

and $-\triangle$ is of class $C^\infty(A)$ with $[iA,-\triangle]=-2(-\triangle)$.

Similarly, one has

$$\mathrm{e}^{-itA}\, M_V\, \mathrm{e}^{itA} = M_{V(\mathrm{e}^t\ \cdot\ )}.$$

Thus, if V is absolutely continuous with $\mathrm{id}_{\mathbb{R}} \cdot V' \in \mathsf{L}^\infty(\mathbb{R})$, one has

$$\text{s-}\frac{\mathrm{d}}{\mathrm{d}t}\, \mathrm{e}^{-itA}\, M_V\, \mathrm{e}^{itA}\Big|_{t=0} = M_{\mathrm{id}_{\mathbb{R}} \cdot V'},$$

and $M_V \in C^1(A)$ with $[iA, M_V] = M_{\mathrm{id}_{\mathbb{R}} \cdot V'}$.

Furthermore, if V' is Dini-continuous, one has $M_V \in C^{1+0}(A)$ since

$$\begin{aligned}
&\int_0^1 \frac{\mathrm{d}t}{t}\, \big\|\, \mathrm{e}^{-itA}[A, M_V]\, \mathrm{e}^{itA} - [A, M_V]\big\|_{\mathscr{B}(\mathcal{H})} \\
&= \int_0^1 \frac{\mathrm{d}t}{t}\, \big\|(\mathrm{id}_{\mathbb{R}} \cdot V')(\mathrm{e}^t\ \cdot\ ) - \mathrm{id}_{\mathbb{R}} \cdot V'\big\|_{\mathsf{L}^\infty(\mathbb{R})} \\
&< \infty.
\end{aligned}$$

We infer that H is of class $C^{1+0}(A)$, with

$$[iH, A] = 2(-\triangle) - M_{\mathrm{id}_{\mathbb{R}} \cdot V'} = 2H - M_{(2V - \mathrm{id}_{\mathbb{R}} \cdot V')}.$$

Now, assume that

$$\lim_{|x| \to \infty} \big(2V - \mathrm{id}_{\mathbb{R}} \cdot V'\big)(x) = 0.$$

Then, a standard result tells us that

$$M_{(2V - \mathrm{id}_{\mathbb{R}} \cdot V')} (-\triangle + i)^{-1} \in \mathscr{K}(\mathcal{H}).$$

(the products $f(Q)\, g(P)$ with $f, g \in C(\mathbb{R})$ vanishing at infinity are compact operators)

Given an open bounded set $I \subset \mathbb{R}$, it follows that

$$
\begin{aligned}
& E^H(I)[iH, A]E^H(I) \\
& = 2E^H(I)HE^H(I) - E^H(I)M_{(2V - \mathrm{id}_{\mathbb{R}} \cdot V')}E^H(I) \\
& \geq 2\inf(I)E^H(I) - E^H(I)M_{(2V - \mathrm{id}_{\mathbb{R}} \cdot V')}(-\triangle + i)^{-1}(-\triangle + i)E^H(I) \\
& = 2\inf(I)E^H(I) + \text{compact operator.}
\end{aligned}
$$

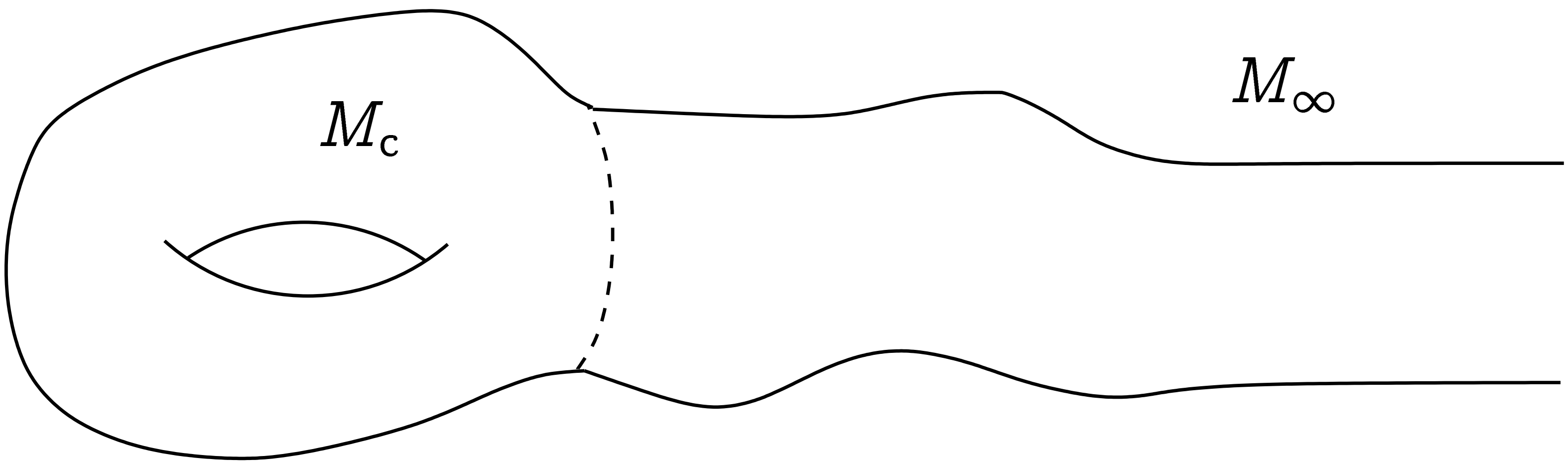
Thus, Theorem 1.17 implies that H has at most finitely many eigenvalues in each open bounded set $I \subset (0, \infty)$ (multiplicities counted), and that H has no singular continuous spectrum in $(0, \infty)$.

(in fact, since $M_V(-\triangle + i)^{-1}$ is compact, one has $\sigma_{\mathrm{ess}}(H) = [0, \infty)$, so that $\sigma_{\mathrm{sc}}(H) = \varnothing$ and $\sigma_{\mathrm{ac}}(H) = [0, \infty)$)

Countless variations/generalisations of this example can be found in the literature:

- the potential V may have singularities (for instance of Coulomb-type)
- the potential V may have anisotropies at infinity
- the Schrödinger operator H may contain a magnetic field
- the Schrödinger operator H can be replaced by an N-body Schrödinger operator
- the Schrödinger operator H can be replaced by a quantum field Hamiltonian
- the Schrödinger operator H can be replaced by a Dirac operator

- the operator $-\triangle$ can be replaced by the Laplace-Beltrami operator (on functions or differential forms) on various types of non-compact manifolds



- the operator $-\triangle$ can be replaced by the combinatorial Laplacian (adjacency matrix) on various types of infinite graphs
- etc . . .

1.5 Time changes of horocycles flows

References:

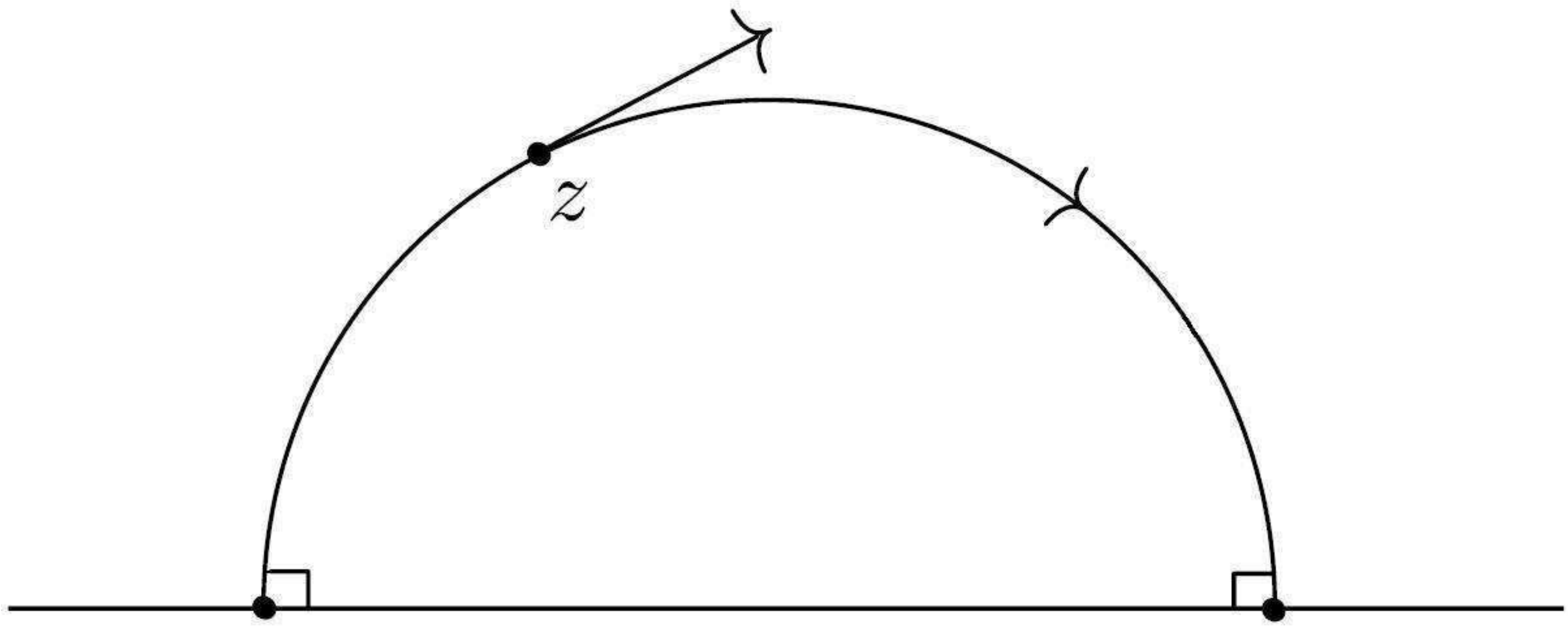
- G. Forni and C. Ulcigrai, Time-changes of horocycle flows, J. Mod. Dyn., 2012
- R. Tiedra, Spectral analysis of time changes of horocycle flows, J. Mod. Dyn., 2012.
- R. Tiedra, Commutator methods for the spectral analysis of uniquely ergodic dynamical systems, preprint on arXiv

Horocycle flow

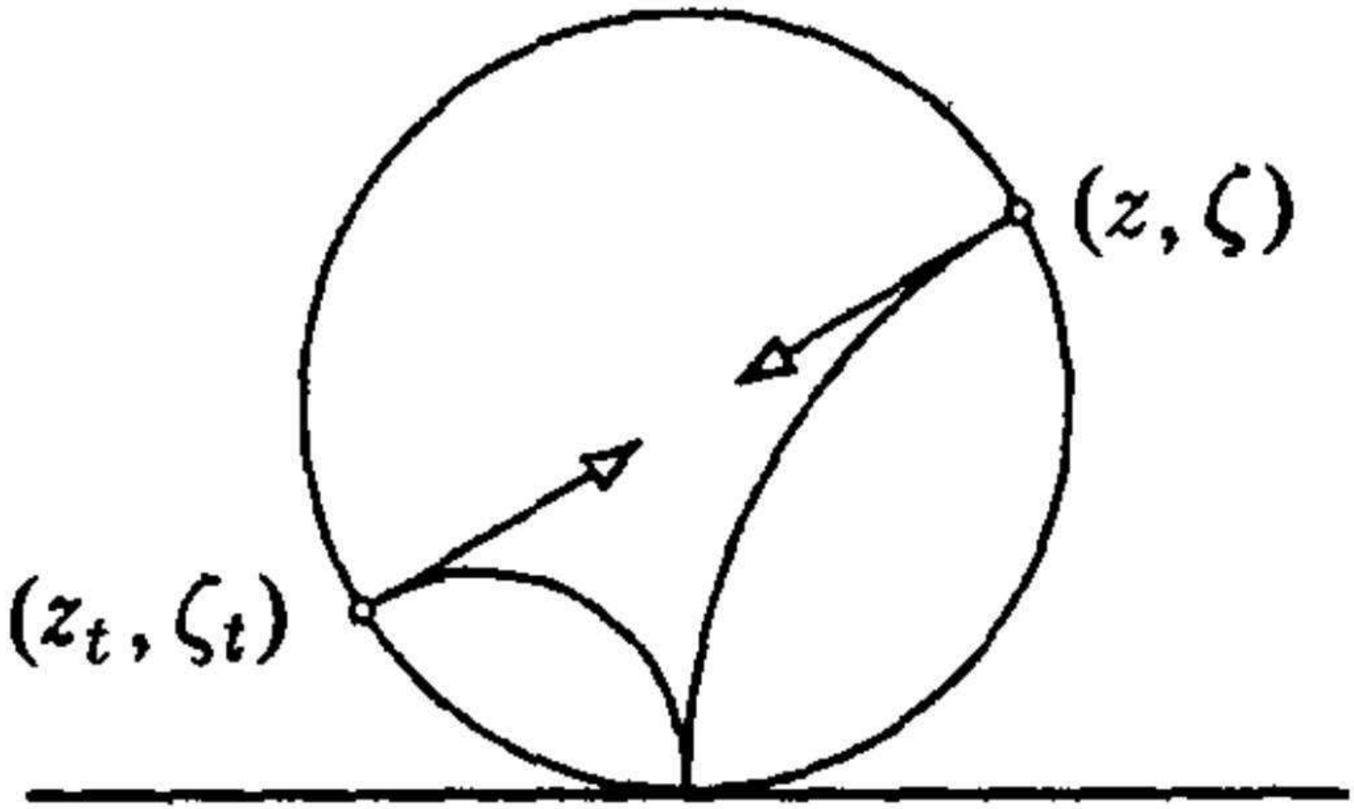
- Σ, compact Riemann surface of genus ≥ 2
- $M := T^1\Sigma$, unit tangent bundle of Σ
- μ_Ω, probability measure on M induced by a volume form Ω

The horocycle flow $\{F_{1,t}\}_{t\in\mathbb{R}}$ and the geodesic flow $\{F_{2,t}\}_{t\in\mathbb{R}}$ are one-parameter groups of diffeomorphisms on M.

Both flows correspond to right translations on M when $M \simeq \Gamma \setminus \mathsf{PSL}(2;\mathbb{R})$, for some cocompact lattice Γ in $\mathsf{PSL}(2;\mathbb{R})$.



Geodesic in the Poincaré half plane



Horocycle flow in the Poincaré half plane

The operators

$$U_j(t)\varphi := \varphi \circ F_{j,t}, \quad t \in \mathbb{R},\ \varphi \in C(M),$$

define strongly continuous unitary groups in $\mathcal{H} := \mathsf{L}^2(M, \mu_\Omega)$ with essentially self-adjoint generators

$$H_j\varphi := -i\mathscr{L}_{X_j}\varphi, \quad \varphi \in C^\infty(M),$$

where X_j is the divergence-free vector field associated with $\{F_{j,t}\}_{t\in\mathbb{R}}$ and $\mathscr{L}_{X_j}$ the corresponding Lie derivative.

The horocycle flow $\{F_{1,t}\}_{t\in\mathbb{R}}$ is uniquely ergodic [Furstenberg 73], mixing of all orders [Marcus 78], and $U_1(t)$ has countable Lebesgue spectrum for each $t \neq 0$ [Parasyuk 53].

The horocycle flow and the geodesic flow satisfy the commutation relation (see for instance [Bachir/Mayer 00])

$$U_2(s)U_1(t)U_2(-s) = U_1(\mathrm{e}^s\, t), \quad s,t \in \mathbb{R}, \tag{1.2}$$

which is a consequence of the matrix identity in $\mathsf{SL}(2,\mathbb{R})$:

$$\begin{pmatrix} \mathrm{e}^{s/2} & 0 \\ 0 & \mathrm{e}^{-s/2} \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-s/2} & 0 \\ 0 & \mathrm{e}^{s/2} \end{pmatrix} = \begin{pmatrix} 1 & \mathrm{e}^s\, t \\ 0 & 1 \end{pmatrix}.$$

Therefore, by applying the strong derivatives $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}$ and $\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}$ to (1.2), one obtains that H_1 is of class $C^\infty(H_2)$ with

$$\big[iH_1, H_2\big] = H_1.$$

Time changes of horocycle flows

Consider a C^1 vector field with the same orientation and proportional to X_1; that is, fX_1 with $f \in C^1\big(M;(0,\infty)\big)$.

The reparametrised time coordinate $h(p,t)$ given by

$$t = \int_0^{h(p,t)} \frac{\mathrm{d}s}{f\big(F_{1,s}(p)\big)}, \quad t \in \mathbb{R},\ p \in M,$$

is such that $h(p,0)=0$, $\lim_{t\to\pm\infty} h(p,t) = \pm\infty$ and $\frac{\mathrm{d}}{\mathrm{d}t} h(p,t) = f\big(F_{1,h(p,t)}(p)\big)$.

The function $\mathbb{R} \ni t \mapsto \widetilde{F}_{1,t}(p) \in M$ given by $\widetilde{F}_{1,t}(p) := F_{1,h(p,t)}(p)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} \widetilde{F}_1(p,t) = (fX_1)_{\widetilde{F}_1(p,t)}, \quad \widetilde{F}_1(p,0) = p,$$

and thus $\{\widetilde{F}_{1,t}\}_{t\in\mathbb{R}}$ is the flow of fX_1.

The operators

$$\widetilde{U}_1(t)\varphi := \varphi \circ \widetilde{F}_{1,t}\,, \quad t \in \mathbb{R},\ \varphi \in C(M),$$

define a strongly continuous unitary group in $\widetilde{\mathcal{H}} := \mathsf{L}^2(M, \mu_\Omega/f)$.

The generator $\widetilde{H} := -i\mathscr{L}_{fX_1}$ of $\{\widetilde{U}_1(t)\}_{t\in\mathbb{R}}$ is essentially self-adjoint on $C^1(M)$ and unitarily equivalent to the operator in $\mathcal{H}$ given by

$$H := f^{1/2} H_1 f^{1/2}.$$

(... the unitary operator $\mathscr{U} : \mathcal{H} \to \widetilde{\mathcal{H}},\ \varphi \mapsto f^{1/2}\varphi$ realises the unitary equivalence ...)

What is the spectral nature of $\widetilde{H}$ (or equivalently of H)?

- Spectral properties are in general not preserved under time changes even though basic ergodic properties are preserved under time changes.
- In 1974, Kushnirenko shows that the flow $\{\widetilde{F}_{1,t}\}_{t\in\mathbb{R}}$ is strongly mixing if f is of class C^∞ and $f - \mathscr{L}_{X_2}(f) > 0$. So, $\widetilde{H}$ has purely continuous spectrum in $\mathbb{R} \setminus \{0\}$ in this case.
- In 2006, Katok and Thouvenot conjecture that $\widetilde{H}$ has absolute continuous spectrum (and even countable Lebesgue spectrum) if f is sufficiently smooth.

Mourre estimate

Let $z \in \mathbb{C} \setminus \mathbb{R}$ and assume for a moment that $f \equiv 1$, so that $H \equiv H_1$. Then, one has $(H+z)^{-1} \in C^1(H_2)$ with

$$\begin{aligned}\big[i(H+z)^{-1}, H_2\big] &= -(H+z)^{-1}[iH, H_2](H+z)^{-1} \\ &= -(H+z)^{-1}H(H+z)^{-1}.\end{aligned}$$

It follows that

$$\begin{aligned}&\big[i\big(H^2+1\big)^{-1}, H_2\big] \\ &= (H+i)^{-1}\big[i(H-i)^{-1}, H_2\big] + \big[i(H+i)^{-1}, H_2\big](H-i)^{-1} \\ &= -\big(H^2+1\big)^{-1}H(H-i)^{-1} - (H+i)^{-1}H\big(H^2+1\big)^{-1} \\ &= -\big(H^2+1\big)^{-1}H(H+i)\big(H^2+1\big)^{-1} - \big(H^2+1\big)^{-1}(H-i)H\big(H^2+1\big)^{-1} \\ &= -\big(H^2+1\big)^{-1}2H^2\big(H^2+1\big)^{-1}.\end{aligned}$$

Thus H^2 is of class $C^\infty(H_2)$ with $[iH^2, H_2] = 2H^2$, and

$$E^{H^2}(I)\big[iH^2, H_2\big]E^{H^2}(I) = E^{H^2}(I)\,2H^2E^{H^2}(I) \geq 2\inf(I)\,E^{H^2}(I)$$

for each open bounded set $I \subset (0,\infty)$.

Therefore, in the case $f \equiv 1$, Mourre's theorem applies to the operator H^2 on the interval $(0,\infty)$.

So, let's try the same approach in the case $f \not\equiv 1\ldots$

If $f \not\equiv 1$, one has $(H+z)^{-1} \in C^1(H_2)$ with

$$\big[i(H+z)^{-1}, H_2\big] = -(H+z)^{-1}(Hg+gH)(H+z)^{-1}$$

and

$$g := \frac{1}{2} - \frac{1}{2}\mathscr{L}_{X_2}\big(\ln(f)\big).$$

$$\text{(note that } g \equiv \frac{f-\mathscr{L}_{X_2}(f)}{2f} > 0 \text{ under Kushnirenko's condition)}$$

A calculation as in the case $f \equiv 1$ shows that

$$\big[i\big(H^2+1\big)^{-1}, H_2\big] = -\big(H^2+1\big)^{-1}\big(H^2g+2HgH+gH^2\big)\big(H^2+1\big)^{-1},$$

which means that $(H^2+1)^{-1} \in C^1(H_2)$ with

$$\big[iH^2, H_2\big] = H^2g+2HgH+gH^2.$$

If $g > 0$ and f is of class C^2, one has

$$
\begin{aligned}
&H^2 g + gH^2 \\
&= \big[H^2, g^{1/2}\big] g^{1/2} + 2\, g^{1/2} H^2 g^{1/2} + g^{1/2}\big[g^{1/2}, H^2\big] \\
&\geq \big[H^2, g^{1/2}\big] g^{1/2} + g^{1/2}\big[g^{1/2}, H^2\big] \\
&= H\big[H, g^{1/2}\big] g^{1/2} + \big[H, g^{1/2}\big] H g^{1/2} + g^{1/2}\big[g^{1/2}, H\big] H + g^{1/2} H \big[g^{1/2}, H\big] \\
&= H\big[H, g^{1/2}\big] g^{1/2} + \big[H, g^{1/2}\big] g^{1/2} H + \big[H, g^{1/2}\big]\big[H, g^{1/2}\big] \\
&\qquad + g^{1/2}\big[g^{1/2}, H\big] H + H g^{1/2}\big[g^{1/2}, H\big] + \big[g^{1/2}, H\big]\big[g^{1/2}, H\big] \\
&= H\big[H, g^{1/2}\big] g^{1/2} + \big[H, g^{1/2}\big] g^{1/2} H + 2\big[H, g^{1/2}\big]^2 + g^{1/2}\big[g^{1/2}, H\big] H \\
&\qquad + H g^{1/2}\big[g^{1/2}, H\big] \\
&= 2\big[H, g^{1/2}\big]^2 \\
&\geq 0.
\end{aligned}
$$

Thus, making everything rigorous, one obtains that

$$\begin{aligned} & E^{H^2}(I)\big[iH^2, H_2\big]E^{H^2}(I) \\ & = E^{H^2}(I)\big(H^2 g + 2HgH + gH^2\big)E^{H^2}(I) \\ & \geq aE^{H^2}(I) \quad \text{with} \quad a := 2\inf(I) \cdot \inf_{p \in M} g(p) > 0 \end{aligned}$$

for each bounded open set $I \subset (0, \infty)$.

Since we also have $(H^2 + 1)^{-1} \in C^2(H_2)$, we conclude by Mourre's theorem that H^2 is purely absolutely continuous outside $\{0\}$, where it has a simple eigenvalue corresponding to the constant functions.

Standard arguments then imply that H has the same spectral properties as H^2.

Summing up:

Theorem 1.18. *Under Kushnirenko's condition and for time changes f of class C^2, the self-adjoint operator $\widetilde{H}$ associated with the vector field fX_1 has purely absolutely continuous spectrum, except at 0, where it has a simple eigenvalue.*

Proof. H and $\widetilde{H}$ are unitarily equivalent. □

In fact, this also holds for noncompact surfaces Σ of finite volume.

Fine, but... Forni and Ulcigrai have obtained the same result (and also Lebesgue maximal spectral type) without assuming Kushnirenko's condition (for compact surfaces and for time changes in a Sobolev space of order $> 11/2$).

So, can we get rid off Kushnirenko's condition ?

Mourre estimate (one more time)

Lemma 1.19 (Conjugate operator). *Let* $f \in C^3\big(M;(0,\infty)\big)$ *and* $L > 0$*. Then, the operator*

$$A_L\,\varphi := \frac{1}{L}\int_0^L \mathrm{d}t\;\mathrm{e}^{itH}\,H_2\,\mathrm{e}^{-itH}\,\varphi, \quad \varphi \in C^1(M),$$

is essentially self-adjoint in $\mathcal{H}$*.*

Idea of the proof. A calculation on $C^1(M)$ shows that

$$\frac{1}{L}\int_0^L \mathrm{d}t\;\mathrm{e}^{itH}\,H_2\,\mathrm{e}^{-itH} = -i\big(\mathscr{L}_X + \tfrac{1}{2}\,\mathrm{div}_\Omega\,X\big),$$

for a certain vector field X on M. Furthermore, if f is of class C^3, then the r.h.s. is the self-adjoint generator of a strongly continuous unitary group (see [Abraham/Marsden 78]). □

(. . . if someone knows how to do it for f of class C^2 . . .)

Replacing H_2 by A_L in the previous calculations and noting that

$$\begin{aligned}\frac{1}{L}\int_0^L \mathrm{d}t\ \mathrm{e}^{itH}\, g\, \mathrm{e}^{-itH} &= \frac{1}{L}\int_0^L \mathrm{d}t\ \mathrm{e}^{it\,\mathscr{U}^*\widetilde{H}\mathscr{U}}\ g\, \mathrm{e}^{-it\,\mathscr{U}^*\widetilde{H}\mathscr{U}}\\ &= \frac{1}{L}\int_0^L \mathrm{d}t\ \mathscr{U}^*\, \mathrm{e}^{it\widetilde{H}}\, g\, \mathrm{e}^{-it\widetilde{H}}\,\mathscr{U}\\ &= \frac{1}{L}\int_0^L \mathrm{d}t\,\big(g\circ \widetilde{F}_{1,-t}\big),\end{aligned}$$

we obtain that $(H^2+1)^{-1}\in C^2(A_L)$ with

$$\big[i(H^2+1)^{-1},A_L\big] = -(H^2+1)^{-1}\big(H^2 g_L+2Hg_LH+g_LH^2\big)(H^2+1)^{-1},$$

where

$$g_L := \frac{1}{L}\int_0^L \mathrm{d}t\,\big(g\circ \widetilde{F}_{1,-t}\big).$$

The flow $\{\widetilde{F}_{1,t}\}_{t\in\mathbb{R}}$ is uniquely ergodic, since it is a reparametrised version of the uniquely ergodic flow $\{F_{1,t}\}_{t\in\mathbb{R}}$ [Humphries 74].

So, the Cesàro mean $g_L = \frac{1}{L}\int_0^L \mathrm{d}t\,\big(g \circ \widetilde{F}_{1,-t}\big)$ converges uniformly on M to $\int_M \mathrm{d}\widetilde{\mu}_\Omega\, g_L$; that is,

$$\begin{aligned}
\lim_{L\to\infty} g_L = \int_M \mathrm{d}\widetilde{\mu}_\Omega\, g_L &= \frac{1}{2} - \frac{1}{2}\int_M \mathrm{d}\widetilde{\mu}_\Omega\, \mathscr{L}_{X_2}\big(\ln(f)\big) \\
&= \frac{1}{2} + \frac{1}{2\int_M f^{-1}\,\mathrm{d}\mu_\Omega}\int_M \mathrm{d}\mu_\Omega\, \mathscr{L}_{X_2}\big(f^{-1}\big) \\
&= \frac{1}{2} + \frac{i}{2\int_M f^{-1}\,\mathrm{d}\mu_\Omega}\big\langle 1, H_2 f^{-1}\big\rangle \\
&= \frac{1}{2}.
\end{aligned}$$

$\implies g_L > 0$ if $L > 0$ is big enough.

So, we got rid off Kushnirenko's condition, and thus have proved the following:

Theorem 1.20. *For time changes f of class C^3, the self-adjoint operator $\widetilde{H}$ associated with the vector field fX_1 has purely absolutely continuous spectrum, except at* 0*, where it has a simple eigenvalue.*

(. . . if someone knows how to prove countable Lebesgue spectrum . . .)

2 Commutator methods for unitary operators

Commutator methods for unitary operators is the unitary analogue of commutator methods for self-adjoint operators.

The theory applies to general unitary operators U (not necessarily of the type e^{iH}), up to the regularity class $C^{1+0}(A)$.

2.1 Unitary operators

A unitary operator U in a Hilbert space $\mathcal{H}$ is a surjective isometry; that is,

$$U^*U = UU^* = 1.$$

Since $U^*U = UU^*$, the spectral theorem for normal operators implies that U admits exactly one complex spectral family E^U with support

$$\operatorname{supp}(E^U) = \sigma(U) \subset \mathbb{S}^1 := \big\{z \in \mathbb{C} \mid |z| = 1\big\}$$

such that

$$U = \int_{\mathbb{C}} z \, E^U(\mathrm{d}z),$$

where $E^U(\lambda + i\mu) := E^{\mathsf{Re}(U)}(\lambda)\, E^{\mathsf{Im}(U)}(\mu)$ for each $\lambda, \mu \in \mathbb{R}$, and

$$\mathsf{Re}(U) := \frac{1}{2}(U + U^*) \qquad \text{and} \qquad \mathsf{Im}(U) := \frac{1}{2i}(U - U^*).$$

One has $U = \int_{\mathbb{R}} \mathrm{e}^{is}\, \widetilde{E}^U(\mathrm{d}s)$ with

$$\widetilde{E}^U(s) := \begin{cases} 0 & \text{if } s < 0 \\ E^U\big(\{\mathrm{e}^{i\tau} \mid \tau \in [0,s]\}\big) & \text{if } s \in [0, 2\pi) \\ 1 & \text{if } s \geq 2\pi. \end{cases}$$

So, one can use the real spectral family $\widetilde{E}^U$ to obtain orthogonal decompositions

$$\mathcal{H} = \mathcal{H}_{\mathrm{p}}(U) \oplus \mathcal{H}_{\mathrm{sc}}(U) \oplus \mathcal{H}_{\mathrm{ac}}(U)$$

$$U = U|_{\mathcal{H}_{\mathrm{p}}(U)} \oplus U|_{\mathcal{H}_{\mathrm{sc}}(U)} \oplus U|_{\mathcal{H}_{\mathrm{ac}}(U)}$$

as in the self-adjoint case.

Example 2.1 (1-parameter groups of unitary operators). *If H is a self-adjoint operator in a Hilbert space $\mathcal{H}$, then*

$$U_t := \mathsf{e}^{-itH}$$

is a unitary operator for each $t \in \mathbb{R}$, and the family $\{U_t\}_{t\in\mathbb{R}}$ defines a strongly continuous 1-parameter group of unitary operators.

Example 2.2 (Koopman operator). *Let $T : X \to X$ be an automorphism of a probability space X with probability measure μ. Then, the Koopman operator U_T in $\mathcal{H} := \mathsf{L}^2(X, \mu)$ given by*

$$U_T : \mathcal{H} \to \mathcal{H}, \quad \varphi \mapsto \varphi \circ T,$$

is a unitary operator.

Ergodicity, weak mixing and strong mixing of an automorphism $T : X \to X$ are expressible in terms of spectral properties of the Koopman operator U_T :

- T is ergodic if and only if 1 is a simple eigenvalue of U_T.
- T is weakly mixing if and only if U_T has purely continuous spectrum in $\{\mathbb{C} \cdot 1\}^\perp$.
- T is strongly mixing if and only if

$$\lim_{n\to\infty} \left\langle \varphi, U_T^n \varphi \right\rangle = 0 \quad \text{for all } \varphi \in \{\mathbb{C} \cdot 1\}^\perp.$$

$$\boxed{\text{strong mixing} \implies \text{weak mixing} \implies \text{ergodicity}}$$

2.2 Commutator methods for unitary operators

References:

- M. A. Astaburuaga, O. Bourget, V. H. Cortés, and C. Fernández, Floquet operators without singular continuous spectrum. J. Funct. Anal., 2006.
- C. Fernández, S. Richard and R. Tiedra, Commutator methods for unitary operators, to appear in J. Spectr. Theory.
- C. R. Putnam, Commutation properties of Hilbert space operators and related topics, Springer-Verlag, 1967.

In [Astaburuaga/Bourget/Cortés/Fernández 06], the authors show an analogue of Mourre's theorem for a unitary operator U in a Hilbert space $\mathcal{H}$.

However . . .

- the regularity assumption is $U \in C^2(A)$,
- the proofs rely once more on differential inequalities for "resolvents" of U.

We want to obtain this result with the weaker assumption $U \in C^{1+0}(A)$ and with a simpler proof !

At the end of the day, we obtain:

Theorem 2.3 (Spectral properties of U). *Let $U \in C^{1+0}(A)$. Assume there exist an open set $\Theta \subset \mathbb{S}^1$, a number $a > 0$ and $K \in \mathscr{K}(\mathcal{H})$ such that*

$$E^U(\Theta)\, U^*[A, U]\, E^U(\Theta) \geq a\, E^U(\Theta) + K.$$

Then, U has at most finitely many eigenvalues in Θ (multiplicities counted), and U has no singular continuous spectrum in Θ.

Sketch of the proof (i)

Why the "commutator" $U^*[A,U]$ is the right expression to consider ?

Imagine that $U = \mathrm{e}^{-iH}$ with $H \in C^1(A)$, then one has

$$
\begin{aligned}
&U^*[A,U] \\
&= i\left(\text{s-}\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{e}^{iH}\,\mathrm{e}^{-itA}\,\mathrm{e}^{-iH}\,\mathrm{e}^{itA}\right)_{t=0} \\
&= i\left(\text{s-}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \mathrm{d}\mu\;\text{s-}\frac{\mathrm{d}}{\mathrm{d}\mu}\,\mathrm{e}^{i\mu H}\,\mathrm{e}^{-itA}\,\mathrm{e}^{-i\mu H}\,\mathrm{e}^{itA}\right)_{t=0} \\
&= -\int_0^1 \mathrm{d}\mu\;\text{s-}\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathrm{e}^{i\mu H}\,H\,\mathrm{e}^{-itA}\,\mathrm{e}^{-i\mu H}\,\mathrm{e}^{itA} - \mathrm{e}^{i\mu H}\,\mathrm{e}^{-itA}\,H\,\mathrm{e}^{-i\mu H}\,\mathrm{e}^{itA}\big)_{t=0} \\
&= -\int_0^1 \mathrm{d}\mu\,\big(\mathrm{e}^{i\mu H}\,H\big[i\,\mathrm{e}^{-i\mu H},A\big] - \mathrm{e}^{i\mu H}\,\big[iH\,\mathrm{e}^{-i\mu H},A\big]\big) \\
&= \int_0^1 \mathrm{d}\mu\;\mathrm{e}^{i\mu H}\,\big[iH,A\big]\,\mathrm{e}^{-i\mu H}\,.
\end{aligned}
$$

Thus,

$$U^*[A,U] = \int_0^1 \mathrm{d}\mu \; \mathrm{e}^{i\mu H} \, [iH, A] \, \mathrm{e}^{-i\mu H},$$

and positivity of $[iH, A]$ leads to positivity of $U^*[A,U]$ and vice versa.

(the idea of using $U^*[A,U]$ dates back to Putnam in the 60's)

Sketch of the proof (ii)

As in the self-adjoint case, one can show a Virial theorem which implies the following:

Corollary 2.4 (Point spectrum of U)**.** *Let U and A be respectively a unitary and a self-adjoint operator in $\mathcal{H}$, with $U \in C^1(A)$. Assume there exist a Borel set $\Theta \subset \mathbb{S}^1$, a number $a > 0$ and $K \in \mathscr{K}(\mathcal{H})$ such that*

$$E^U(\Theta)\, U^*[A, U]\, E^U(\Theta) \geq a E^U(\Theta) + K.$$

Then, U has at most finitely many eigenvalues in Θ (multiplicities counted).

If $U \in C^1(A)$ and

$$E^U(\Theta)\,U^*[A,U]\,E^U(\Theta) \geq a\,E^U(\Theta) + K,$$

then the corollary implies that U has at most finitely many eigenvalues in Θ (multiplicities counted).

So, there exists $\theta \in \Theta$ which is not an eigenvalue of U, and the range $\mathrm{Ran}(1-\bar{\theta}U)$ of $1-\bar{\theta}U$ is dense in $\mathcal{H}$.

Indeed, if $\psi \in \mathcal{H}$ is such that $\psi \perp \mathrm{Ran}(1-\bar{\theta}U)$, then

$$\begin{aligned}\big\langle \psi, (1-\bar{\theta}U)\varphi\big\rangle = 0 \quad \text{for all } \varphi \in \mathcal{H} &\implies (1-\theta U^*)\psi = 0\\ &\implies U\psi = \theta\psi\\ &\implies \psi = 0.\end{aligned}$$

Furthermore, the Cayley transform of U at the point θ; that is, the operator

$$H_\theta := -i\big(1+\bar\theta U\big)\big(1-\bar\theta U\big)^{-1}, \quad \mathcal{D}(H_\theta) := \mathsf{Ran}(1-\bar\theta U),$$

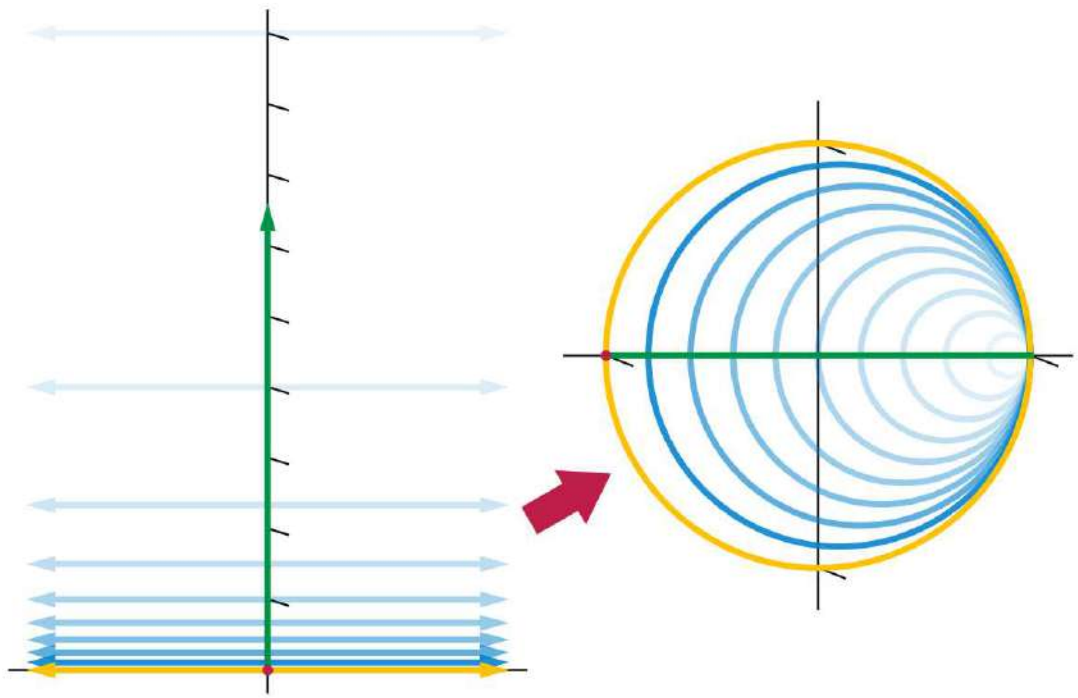
is self-adjoint.

Indeed, H_θ is self-adjoint if and only if

$$\begin{aligned}
&\mathsf{Ran}(H_\theta + i) = \mathsf{Ran}(H_\theta - i) = \mathcal{H}\\
\iff\ &\mathsf{Ran}\left(-2i\bar\theta U(1-\bar\theta U)^{-1}|_{\mathsf{Ran}(1-\bar\theta U)}\right)\\
&\qquad\qquad = \mathsf{Ran}\left(-2i(1-\bar\theta U)^{-1}|_{\mathsf{Ran}(1-\bar\theta U)}\right) = \mathcal{H}\\
\iff\ &-2i\bar\theta U\,\mathcal{H} = -2i\,\mathcal{H} = \mathcal{H}\\
\iff\ &\mathcal{H} = \mathcal{H} = \mathcal{H}.
\end{aligned}$$

For any Borel set $\Theta \subset \mathbb{S}^1$, the spectral measure E^{H_θ} of H_θ satisfies

$$E^{H_\theta}(I) = E^U(\Theta) \qquad \text{with} \qquad I := \left\{ -i\,\frac{1+\bar{\theta}z}{1-\bar{\theta}z} \mid z \in \Theta \right\}.$$



Cayley transform of $\mathbb{R}$ (for $\theta = -i$)

Sketch of the proof (iii)

One has

$$\begin{aligned}(H_\theta - i)^{-1} &= \Big\{\big(-i(1+\bar{\theta}U) - i(1-\bar{\theta}U)\big)(1-\bar{\theta}U)^{-1}\Big\}^{-1} \\ &= \Big\{-2i\big(1-\bar{\theta}U\big)^{-1}\Big\}^{-1} \\ &= -\frac{1}{2i}\big(1-\bar{\theta}U\big).\end{aligned}$$

Thus,

$$\big[A,(H_\theta - i)^{-1}\big] = \left[A - \frac{1}{2i}\big(1-\bar{\theta}U\big)\right] = \frac{\bar{\theta}}{2i}\big[A,U\big],$$

and the regularity condition $U \in C^{1+0}(A)$ implies the regularity condition $(H_\theta - i)^{-1} \in C^{1+0}(A)$.

Sketch of the proof (iv)

A calculation in $\mathscr{B}\big(\mathcal{D}(H_\theta),\mathcal{D}(H_\theta)^*\big)$ shows that

$$\begin{aligned}
[iH_\theta,A] &= \Big[\big(1+\bar\theta U\big)\big(1-\bar\theta U\big)^{-1},A\Big]\\
&= \big(1+\bar\theta U\big)\Big[\big(1-\bar\theta U\big)^{-1},A\Big]+\Big[\big(1+\bar\theta U\big),A\Big]\big(1-\bar\theta U\big)^{-1}\\
&\ \ \vdots\\
&= 2\big\{(1-\bar\theta U)^{-1}\big\}^*U^*[A,U](1-\bar\theta U)^{-1}
\end{aligned}$$

So, the positivity of $U^*[A,U]$ on a Borel set $\Theta\subset\mathbb{S}^1$ implies the positivity of $[iH_\theta,A]$ on the corresponding set $I\subset\mathbb{R}$.

Since H_θ is of class $C^{1+0}(A)$, the usual (self-adjoint) Mourre's theorem implies that H_θ has no singular continuous spectrum in I.

Now, suppose by absurd that U has some singular continuous spectrum in $\Theta \setminus \{\theta\}$. Then, there exist $\varphi \in \mathcal{H} \setminus \{0\}$ and $\mathcal{V} \subset [0, 2\pi)$ such that

$$\text{closure}\big(\mathrm{e}^{i\mathcal{V}}\big) \subset \Theta \setminus \{\theta\}, \quad |\mathcal{V}| = 0 \quad \text{and} \quad \widetilde{E}^U(\mathcal{V})\varphi = \varphi.$$

This implies that

$$\widetilde{E}^U(\mathcal{V})\varphi = \varphi \iff E^U(\mathrm{e}^{i\mathcal{V}})\varphi = \varphi \iff E^{H_\theta}(J)\varphi = \varphi,$$

with

$$J := \left\{ -i\,\frac{1 + \bar{\theta}\,\mathrm{e}^{iv}}{1 - \bar{\theta}\,\mathrm{e}^{iv}} \mid v \in \mathcal{V} \right\} \subset I.$$

But, the function

$$\mathcal{V} \ni v \mapsto -i\,\frac{1+\bar{\theta}\,\mathrm{e}^{iv}}{1-\bar{\theta}\,\mathrm{e}^{iv}} \in J$$

has the Luzin N property. So $|J| = 0$, and thus $\varphi = 0$ since H_θ has no singular continuous spectrum in $J \subset I$.

Since $\varphi \in \mathcal{H} \setminus \{0\}$, this is a contradiction. So, U has no singular continuous spectrum in $\Theta \setminus \{\theta\}$, and thus no singular continuous spectrum in Θ.

No need to re-do any proof with differential inequalities. We just used the Cayley transform and the pre-existing self-adjoint theory.

We also have the following perturbation result:

Corollary 2.5 (Perturbations of U)**.** *Let U, V be unitary, with $U, V \in C^{1+0}(A)$. Assume there exist an open set $\Theta \subset \mathbb{S}^1$, a number $a > 0$ and $K \in \mathscr{K}(\mathcal{H})$ such that*

$$E^U(\Theta) U^* [A, U] E^U(\Theta) \geq a E^U(\Theta) + K. \tag{2.1}$$

Suppose also that $(V - 1) \in \mathscr{K}(\mathcal{H})$ is compact. Then, VU has at most finitely many eigenvalues in each closed subset of Θ (multiplicities counted), and VU has no singular continuous spectrum in Θ.

- the Mourre estimate (2.1) depends on U only (V is the perturbation)
- UV and VU are unitarily equivalent since $UV = U(VU)U^*$

2.3 Perturbations of bilateral shifts

Let U be a bilateral shift on a Hilbert space $\mathcal{H}$ with wandering subspace $\mathcal{M} \subset \mathcal{H}$, *i.e.*,

$$\mathcal{M} \perp U^n(\mathcal{M}) \ \text{ for each } n \in \mathbb{Z} \setminus \{0\} \quad \text{and} \quad \mathcal{H} = \bigoplus_{n \in \mathbb{Z}} U^n(\mathcal{M}).$$

Using the notation $\varphi \equiv \{\varphi_n\} \in \mathcal{H}$, define the (number) operator

$$A\varphi := \{n\varphi_n\}, \quad \varphi \in \mathcal{D}(A) := \left\{\psi \in \mathcal{H} \mid \textstyle\sum_{n\in\mathbb{Z}} n^2 \|\psi_n\|^2 < \infty\right\},$$

which is self-adjoint since it is a maximal multiplication operator in a ℓ^2-space.

One has for each $\varphi \in \mathcal{D}(A)$

$$\begin{aligned}\big\langle A\varphi, U\varphi\big\rangle - \big\langle \varphi, UA\varphi\big\rangle &= \big\langle \{n\varphi_n\}, \{\varphi_{n+1}\}\big\rangle - \big\langle \{\varphi_n\}, U\{n\varphi_n\}\big\rangle \\ &= \big\langle \{\varphi_n\}, \{(n+1)\varphi_{n+1}\}\big\rangle - \big\langle \{\varphi_n\}, \{n\varphi_{n+1}\}\big\rangle \\ &= \big\langle \varphi, U\varphi\big\rangle,\end{aligned}$$

meaning that $U \in C^\infty(A) \subset C^{1+0}(A)$ with $U^*[A,U] = U^*U = 1$.

Thus, Theorem 2.3 implies that U has purely absolutely continuous spectrum, as it is well known.

In fact, the conditions $U \in C^1(A)$ and $[A,U] = U$ imply that

$$\text{s-}\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{e}^{-itA}\,U\,\mathrm{e}^{itA} = -i\,\mathrm{e}^{-itA}\,U\,\mathrm{e}^{itA} \quad\Longleftrightarrow\quad \mathrm{e}^{-itA}\,U\,\mathrm{e}^{itA} = \mathrm{e}^{-it}\,U.$$

So, U is unitarily equivalent to $\mathrm{e}^{-it}\,U$ for each $t \in \mathbb{R}$, and thus has purely Lebesgue spectrum covering the whole circle $\mathbb{S}^1$.

Let V be another unitary operator with $V \in C^{1+0}(A)$ and $(V - 1) \in \mathscr{K}(\mathcal{H})$.

We deduce from Corollary 2.5 that VU has purely absolutely continuous spectrum except, possibly, at a finite number of points of $\mathbb{S}^1$, where VU may have eigenvalues of finite multiplicity.

2.4 Perturbations of the Schrödinger free evolution

The Schrödinger free evolution $\{U_t\}_{t\in\mathbb{R}}$ in $\mathcal{H} := \mathsf{L}^2(\mathbb{R}^d)$ given by

$$U_t := \mathrm{e}^{-itP^2}, \quad t \in \mathbb{R},$$

satisfies

$$\sigma(U_t) = \sigma_{\mathrm{ac}}(U_t) = \mathbb{S}^1 \quad \text{for each } t \neq 0.$$

Indeed, one has for each $s \in [0, 2\pi)$ and $t \neq 0$ that

$$\begin{aligned} E^{\,\mathrm{e}^{-itP^2}}(\mathrm{e}^{is}) &= E^{\cos(-tP^2)}\big(\cos(s)\big)\, E^{\sin(-tP^2)}\big(\sin(s)\big) \\ &= E^{-tP^2}\big([0,s] + 2\pi\mathbb{Z}\big)\, E^{-tP^2}\big([0,s] + 2\pi\mathbb{Z}\big) \\ &= E^{P^2}\left([0,-s/t] + \frac{2\pi}{t}\mathbb{Z}\right). \end{aligned}$$

What can we say about perturbations of the type VU_t ?

The operator

$$A := \frac{1}{2}\left\{(P^2+1)^{-1}P\cdot Q + Q\cdot P(P^2+1)^{-1}\right\}$$

is essentially self-adjoint on $C_{\mathrm{c}}^\infty(\mathbb{R}^d)$ (because the vector field $X_x := x(x^2+1)^{-1} \in \mathbb{R}^d$ is complete), and calculations on $C_{\mathrm{c}}^\infty(\mathbb{R}^d)$ show that $U_t \in C^1(A)$ with

$$\begin{aligned}
&(U_t)^*[A, U_t] \\
&= \frac{1}{2}\,\mathrm{e}^{itP^2}\sum_j \Big\{(P^2+1)^{-1}P_j\big[Q_j, \mathrm{e}^{-itP^2}\big] + \big[Q_j, \mathrm{e}^{-itP^2}\big]P_j(P^2+1)^{-1}\Big\} \\
&= t\,\mathrm{e}^{itP^2}\sum_j \Big\{(P^2+1)^{-1}P_j^2\,\mathrm{e}^{-itP^2} + \mathrm{e}^{-itP^2}\,P_j^2(P^2+1)^{-1}\Big\} \\
&= 2tP^2(P^2+1)^{-1}.
\end{aligned}$$

Further commutations on $C_{\rm c}^{\infty}(\mathbb{R}^d)$ show that $U_t \in C^2(A)$.

Moreover, if $t > 0$ and closure$(\Theta) \cap \{1\} = \varnothing$, there exists $\delta > 0$ such that

$$E^{U_t}(\Theta)(U_t)^*[A, U_t]\, E^{U_t}(\Theta) \geq 2t\delta(\delta+1)^{-1} E^{U_t}(\Theta).$$

So, all the assumptions for U_t are satisfied, and we have:

Lemma 2.6. *If $V \in C^{1+0}(A)$ and $(V-1) \in \mathscr{K}(\mathcal{H})$, then the eigenvalues of VU_t outside $\{1\}$ are of finite multiplicity and can accumulate only at $\{1\}$. Furthermore, VU has no singular continuous spectrum.*

This extends previous results on the Schrödinger free evolution perturbed by "periodic kicks" ($V = \mathrm{e}^{iB}$ with $B = B^*$ of finite rank).

2.5 Skew products over translations

Let $\{y_t\}_{t\in\mathbb{R}}$ be a C^1 one-parameter subgroup of a compact metric abelian Banach Lie group X with normalised Haar measure μ (such group X is isomorphic to a subgroup of $\mathbb{T}^{\aleph_0} \equiv (\mathbb{R}/\mathbb{Z})^{\aleph_0}$).

Let $\{F_t\}_{t\in\mathbb{R}}$ be the corresponding translation flow,

$$F_t(x) := y_t\, x, \quad t \in \mathbb{R},\ x \in X,$$

and let $\{V_t\}_{t\in\mathbb{R}}$ the corresponding strongly continuous unitary group in $\mathcal{H} := \mathsf{L}^2(X,\mu)$,

$$V_t\varphi := \varphi \circ F_t, \quad t \in \mathbb{R},\ \varphi \in C(X).$$

The generator H of $\{V_t\}_{t\in\mathbb{R}}$ given by

$$H\varphi := -i\mathscr{L}_Y\varphi, \quad \varphi \in C^\infty(X),$$

with Y the vector field associated with $\{F_t\}_{t\in\mathbb{R}}$ and $\mathscr{L}_Y$ the corresponding Lie derivative, is essentially self-adjoint on $C^\infty(X)$.

Let G be a compact metric abelian group with Haar measure ν and character group $\widehat{G}$, and let $\phi : X \to G$ be a measurable function (cocycle).

We want to apply commutator methods to the Koopman operator

$$W\psi := \psi \circ T\,, \quad \psi \in \mathsf{L}^2(X \times G, \mu \times \nu),$$

with T the (measure-preserving invertible) skew product

$$T : X \times G \to X \times G, \quad (x, z) \mapsto \big(y_1\,x, \phi(x)\,z\big).$$

The operator W is reduced by the orthogonal decomposition (given by the Peter-Weyl theorem)

$$\mathsf{L}^2(X \times G, \mu \times \nu) = \bigoplus_{\chi \in \widehat{G}} L_\chi, \quad L_\chi := \{\varphi \otimes \chi \mid \varphi \in \mathcal{H}\},$$

and $W|_{L_\chi}$ is unitarily equivalent to the unitary operator

$$U_\chi \varphi := (\chi \circ \phi) V_1 \varphi, \quad \varphi \in \mathcal{H}.$$

Furthermore, the operator U_χ satisfies the following purity law:

If F_1 is ergodic, the spectrum of U_χ has uniform multiplicity and is either purely punctual, purely singular continuous or purely Lebesgue (see [Helson 86] in the case $X = G = \mathbb{T}$).

We assume the following:

Assumption 2.7. *The translation* F_1 *is ergodic and* $\phi : X \to G$ *satisfies* $\phi = \xi\eta$, *where*

(i) $\xi : X \to G$ *is a continuous group homomorphism,*

(ii) $\eta \in C(X;G)$ *has a Lie derivative* $\mathscr{L}_Y(\chi \circ \eta)$ *which satisfies*

$$\int_0^1 \frac{\mathrm{d}t}{t} \left\| \mathscr{L}_Y(\chi \circ \eta) \circ F_t - \mathscr{L}_Y(\chi \circ \eta) \right\|_{\mathsf{L}^\infty(X)} < \infty.$$

Two comments:

- $\chi \circ \xi$ encodes the "topological degree" of the cocycle $\chi \circ \phi$.
- (ii) means that $\mathscr{L}_Y(\chi \circ \eta)$ is of Dini-type along the translation flow $\{F_t\}_{t \in \mathbb{R}}$.

Define

$$\xi_0 := \frac{\mathrm{d}}{\mathrm{d}t}(\chi\circ\xi)(y_t)\Big|_{t=0}, \quad g := |\xi_0|^2 - \xi_0\,\frac{\mathscr{L}_Y(\chi\circ\eta)}{\chi\circ\eta} \quad \text{and} \quad A := -i\,\xi_0 H,$$

and observe that $g : X \to \mathbb{R}$ is of Dini-type along $\{F_t\}_{t\in\mathbb{R}}$ and that A is self-adjoint with $\mathcal{D}(A) \supset \mathcal{D}(H)$.

Since A and V_1 commute, we have for each $\varphi \in C^\infty(X)$ that

$$\begin{aligned}\langle A\varphi, U_\chi\varphi\rangle - \langle\varphi, U_\chi A\varphi\rangle &= \langle\varphi, [A, \chi\circ\phi]V_1\varphi\rangle \\ &= \langle\varphi, -\xi_0\mathscr{L}_Y(\chi\circ\phi)V_1\varphi\rangle,\end{aligned}$$

with

$$\begin{aligned}\mathscr{L}_Y(\chi\circ\phi) &= \mathscr{L}_Y(\chi\circ\xi)(\chi\circ\eta) + (\chi\circ\xi)\mathscr{L}_Y(\chi\circ\eta) \\ &= \left(\xi_0 + \frac{\mathscr{L}_Y(\chi\circ\eta)}{\chi\circ\eta}\right)(\chi\circ\phi).\end{aligned}$$

It follows that

$$\big\langle A\varphi, U_\chi\varphi\big\rangle - \big\langle\varphi, U_\chi A\varphi\big\rangle = \big\langle\varphi, gU_\chi\varphi\big\rangle,$$

with $g \in \mathsf{L}^\infty(X)$. So, one has $U_\chi \in C^1(A)$ with $\big[A, U_\chi\big] = gU_\chi$ due to the density of $C^\infty(X)$ in $\mathcal{D}(A)$.

Since g is of Dini-type along $\{F_t\}_{t\in\mathbb{R}}$, the equalities

$$\begin{aligned}
&\int_0^1 \frac{\mathrm{d}t}{t}\,\big\|\,\mathrm{e}^{-itA}\,\big[A, U_\chi\big]\,\mathrm{e}^{itA} - \big[A, U_\chi\big]\big\|_{\mathscr{B}(\mathcal{H})}\\
&= \int_0^1 \frac{\mathrm{d}t}{t}\,\big\|\,\mathrm{e}^{-itA}\,gU_\chi\,\mathrm{e}^{itA} - gU_\chi\big\|_{\mathscr{B}(\mathcal{H})}\\
&= \int_0^1 \frac{\mathrm{d}t}{t}\,\big\|\big(\mathrm{e}^{-itA}\,g\,\mathrm{e}^{itA} - g\big)\,\mathrm{e}^{-itA}\,U_\chi\,\mathrm{e}^{itA} + g\big(\mathrm{e}^{-itA}\,U_\chi\,\mathrm{e}^{itA} - U_\chi\big)\big\|_{\mathscr{B}(\mathcal{H})}
\end{aligned}$$

imply that $U_\chi \in C^{1+0}(A)$.

If the function g were strictly positive, we would be able to apply Theorem 2.3 since

$$(U_\chi)^* \left[A, U_\chi\right] = (U_\chi)^* g U_\chi \geq \inf_{x \in X} g(x) > 0.$$

But, this is a priori not the case since

$$g = |\xi_0|^2 - \xi_0 \frac{\mathscr{L}_Y(\chi \circ \eta)}{\chi \circ \eta} \equiv \text{positive constant} + \text{total derivative.}$$

Nonetheless, the same averaging of the conjugate operator A as the one used for horocycle flows may work and lead to a strictly positive function g.

Since $U_\chi \in C^1(A)$, we have $U_\chi^\ell \in C^1(A)$ and $U_\chi^\ell \mathcal{D}(A) = \mathcal{D}(A)$ for each $\ell \in \mathbb{Z}$, and thus the operator

$$A_n \varphi := \frac{1}{n} \sum_{\ell=0}^{n-1} U_\chi^{-\ell} A U_\chi^\ell \varphi = \frac{1}{n} \sum_{\ell=0}^{n-1} U_\chi^{-\ell} \big[A, U_\chi^\ell\big] \varphi + A\varphi, \quad \varphi \in \mathcal{D}(A_n) := \mathcal{D}(A),$$

is self-adjoint since $\frac{1}{n} \sum_{\ell=0}^{n-1} U_\chi^{-\ell} \big[A, U_\chi^\ell\big]$ is bounded.

Doing the same calculations as before with A_n instead of A, one obtains that $U_\chi \in C^{1+0}(A_n)$ with

$$\big[A_n, U_\chi\big] = \frac{1}{n} \sum_{\ell=0}^{n-1} U_\chi^{-\ell} \big[A, U_\chi\big] U_\chi^\ell = \frac{1}{n} \sum_{\ell=0}^{n-1} U_\chi^{-\ell} (g U_\chi) U_\chi^\ell = g_n U_\chi$$

and

$$g_n := \left(\frac{1}{n} \sum_{\ell=0}^{n-1} U_\chi^{-\ell} g U_\chi^\ell \right) = \frac{1}{n} \sum_{\ell=0}^{n-1} g \circ F_{-\ell}.$$

Since F_1 is ergodic, we know (see [Cornfeld/Fomin/Sinaĭ 82]) that the flow $\{F_\ell\}_{\ell\in\mathbb{Z}}$ is uniquely ergodic and that

$$\xi_0 = \frac{\mathrm{d}}{\mathrm{d}t}(\chi\circ\xi)(y_t)\Big|_{t=0} \neq 0 \quad \text{if} \quad \chi\circ\xi \not\equiv 1.$$

Using the notation $\chi\circ\eta = \mathrm{e}^{i f_{\chi,\eta}}$, we infer that

$$\begin{aligned}\lim_{n\to\infty} g_n = \int_X \mathrm{d}\mu\, g &= |\xi_0|^2 - \xi_0 \int_X \mathrm{d}\mu\, \frac{\mathscr{L}_Y(\chi\circ\eta)}{\chi\circ\eta} \\ &= |\xi_0|^2 + \xi_0 \left\langle 1, H f_{\chi,\eta}\right\rangle \\ &= |\xi_0|^2\end{aligned}$$

uniformly on X.

Thus, $g_n > 0$ if n is large enough, and

$$(U_\chi)^* \left[A_n, U_\chi\right] = (U_\chi)^* g_n U_\chi \geq \inf_{x \in X} g_n(x) > 0$$

as desired.

Putting everything together, we obtain the following:

Theorem 2.8 (Spectral properties of W). *Let F_1 be ergodic and let ϕ satisfy Assumption 2.7 with $\chi \circ \xi \not\equiv 1$. Then, U_χ has purely Lebesgue spectrum. In particular, the restriction of W to the subspace $\bigoplus_{\chi \in \widehat{G},\, \chi \circ \xi \not\equiv 1} L_\chi \subset \mathsf{L}^2(X \times G, \mu \times \nu)$ has countable Lebesgue spectrum.*

Two remarks:

- In the case $X = \mathbb{T}^d$, $G = \mathbb{T}^{d'}$ with $d, d' \geq 1$, this complements previous results of [Iwanik/Lemańczyk/Rudolph 93-99], where $\mathscr{L}_Y(\chi \circ \eta)$ is of bounded variation instead of Dini-type.

  (bounded variation and Dini-continuity are mutually independent)

- If we do not assume that $\mathscr{L}_Y(\chi \circ \eta)$ is of Dini-type, we can already infer that W has purely continuous spectrum in $\bigoplus_{\chi \in \widehat{G},\, \chi \circ \xi \not\equiv 1} L_\chi$ due to the corollary on the point spectrum (Corollary 2.4) and the purity law.