A formula relating localisation observables to the variation of energy in Hamiltonian dynamics

Rafael Tiedra
(Pontificia Universidad Católica de Chile)

Pucón, December 2010

Joint work with Antoine Gournay (Neuchâtel)

1 Motivation

1 Motivation

Various papers (some with C. Gérard '07 and S. Richard '09-10) on quantum scattering theory put into evidence a useful formula relating time evolution to localisation observables.

1 Motivation

Various papers (some with C. Gérard '07 and S. Richard '09-10) on quantum scattering theory put into evidence a useful formula relating time evolution to localisation observables.

Though simple, this formula does not seem to be known in classical mechanics... (see however [Buslaev/Pushnitski 10])

1 Motivation

Various papers (some with C. Gérard '07 and S. Richard '09-10) on quantum scattering theory put into evidence a useful formula relating time evolution to localisation observables.

Though simple, this formula does not seem to be known in classical mechanics... (see however [Buslaev/Pushnitski 10])

This is the topic of this talk.

2 Guiding example

2 Guiding example

Consider the symplectic manifold $M := T^*\mathbb{R}^d \simeq \mathbb{R}^{2d}$ with coordinates (q, p), 2-form $\omega := \sum_j dq^j \wedge dp_j$ and Poisson bracket $\{\cdot, \cdot\}$.

2 Guiding example

Consider the symplectic manifold $M := T^*\mathbb{R}^d \simeq \mathbb{R}^{2d}$ with coordinates (q, p), 2-form $\omega := \sum_j dq^j \wedge dp_j$ and Poisson bracket $\{\cdot, \cdot\}$.

Let $H(q, p) := h(p)$ be a kinetic energy Hamiltonian with (complete) flow $\{\varphi_t\}_{t\in\mathbb{R}}$, let $\Phi(p, q) := q$ the position observable and let χ_1 the characteristic function for $B_1 := \{x \in \mathbb{R}^d \mid |x| \leq 1\}$.

2 Guiding example

Consider the symplectic manifold $M := T^*\mathbb{R}^d \simeq \mathbb{R}^{2d}$ with coordinates (q, p), 2-form $\omega := \sum_j dq^j \wedge dp_j$ and Poisson bracket $\{\cdot, \cdot\}$.

Let $H(q, p) := h(p)$ be a kinetic energy Hamiltonian with (complete) flow $\{\varphi_t\}_{t\in\mathbb{R}}$, let $\Phi(p, q) := q$ the position observable and let χ_1 the characteristic function for $B_1 := \{x \in \mathbb{R}^d \mid |x| \leq 1\}$.

Then one has for $(p, q) \in M$ with $(\nabla h)(p) \neq 0$

$$\lim_{r\to\infty} \frac{1}{2}\int_0^\infty dt \left[\big(\chi_1(\Phi/r)\circ\varphi_{-t}\big)(p, q) - \big(\chi_1(\Phi/r)\circ\varphi_t\big)(p, q)\right] = T(p, q)\,,$$

where $T(p, q) = \dfrac{q \cdot (\nabla h)(p)}{(\nabla h)(p)^2}$

2 Guiding example

Consider the symplectic manifold $M := T^*\mathbb{R}^d \simeq \mathbb{R}^{2d}$ with coordinates (q,p), 2-form $\omega := \sum_j dq^j \wedge dp_j$ and Poisson bracket $\{\cdot,\cdot\}$.

Let $H(q,p) := h(p)$ be a kinetic energy Hamiltonian with (complete) flow $\{\varphi_t\}_{t\in\mathbb{R}}$, let $\Phi(p,q) := q$ the position observable and let χ_1 the characteristic function for $B_1 := \{x \in \mathbb{R}^d \mid |x| \leq 1\}$.

Then one has for $(p,q) \in M$ with $(\nabla h)(p) \neq 0$

$$\lim_{r\to\infty} \frac{1}{2}\int_0^\infty dt \left[\big(\chi_1(\Phi/r)\circ\varphi_{-t}\big)(p,q) - \big(\chi_1(\Phi/r)\circ\varphi_t\big)(p,q)\right] = T(p,q)\,,$$

where $T(p,q) = \dfrac{q\cdot(\nabla h)(p)}{(\nabla h)(p)^2} \propto \dfrac{\text{length} \times \text{velocity}}{\text{velocity}^2} \propto \text{time}.$

- For $r > 0$ fixed, the l.h.s. is equal to the difference of times spent by the orbit $\{\varphi_t(p, q)\}_{t\in\mathbb{R}}$ in the past and in the future within $B_r := \{x \in \mathbb{R}^d \mid |x| \leq r\}$.

- For $r > 0$ fixed, the l.h.s. is equal to the difference of times spent by the orbit $\{\varphi_t(p,q)\}_{t\in\mathbb{R}}$ in the past and in the future within $B_r := \{x \in \mathbb{R}^d \mid |x| \leq r\}$.

- The map $\frac{d}{dH} := \{T, \cdot\}$ is a derivation on $C^\infty(M)$, so T can be seen as an observable "derivative with respect to the energy H" on M, since

$$\frac{d}{dH}(H) \equiv \{T, H\} = \left\{\frac{q \cdot (\nabla h)(p)}{(\nabla h)(p)^2}, h(p)\right\} = \sum_j \{q_j, h(p)\} \frac{(\partial_j h)(p)}{(\nabla h)(p)^2} = 1.$$

- For $r > 0$ fixed, the l.h.s. is equal to the difference of times spent by the orbit $\{\varphi_t(p,q)\}_{t\in\mathbb{R}}$ in the past and in the future within $B_r := \{x \in \mathbb{R}^d \mid |x| \leq r\}$.

- The map $\frac{d}{dH} := \{T, \cdot\}$ is a derivation on $C^\infty(M)$, so T can be seen as an observable "derivative with respect to the energy H" on M, since

$$\frac{d}{dH}(H) \equiv \{T, H\} = \left\{\frac{q \cdot (\nabla h)(p)}{(\nabla h)(p)^2}, h(p)\right\} = \sum_j \{q_j, h(p)\} \frac{(\partial_j h)(p)}{(\nabla h)(p)^2} = 1.$$

The formula provides a relation between sojourn times and variation of energy along classical orbits.

The formula can be extended to abstract Hamiltonians H and abstract position observables Φ on a symplectic manifold M, if H and Φ satisfy an appropriate "commutation" relation.

3 Framework

3 Framework

- M, finite or infinite-dimensional symplectic manifold with symplectic 2-form ω and Poisson bracket $\{\cdot, \cdot\}$

3 Framework

- M, finite or infinite-dimensional symplectic manifold with symplectic 2-form ω and Poisson bracket $\{\cdot, \cdot\}$
- $H \in C^\infty(M)$, Hamiltonian on M with vector field X_H and complete flow $\{\varphi_t\}_{t\in\mathbb{R}}$

3 Framework

- M, finite or infinite-dimensional symplectic manifold with symplectic 2-form ω and Poisson bracket $\{\cdot, \cdot\}$
- $H \in C^\infty(M)$, Hamiltonian on M with vector field X_H and complete flow $\{\varphi_t\}_{t\in\mathbb{R}}$
- $\Phi \equiv (\Phi_1, \ldots, \Phi_d) \in C^\infty(M; \mathbb{R}^d)$, a family of observables.

3 Framework

- M, finite or infinite-dimensional symplectic manifold with symplectic 2-form ω and Poisson bracket $\{\cdot, \cdot\}$
- $H \in C^\infty(M)$, Hamiltonian on M with vector field X_H and complete flow $\{\varphi_t\}_{t\in\mathbb{R}}$
- $\Phi \equiv (\Phi_1, \ldots, \Phi_d) \in C^\infty(M; \mathbb{R}^d)$, a family of observables.

Assumption:

$\big\{\{\Phi_j, H\}, H\big\} = 0$ for each j.

4 Main theorem

Under the Assumption, we have:

Theorem 4.1. *There exist a closed subset* $\mathsf{Crit}(H, \Phi) \subset M$ *and an observable* $T \in C^\infty\big(M \setminus \mathsf{Crit}(H, \Phi)\big)$ *satisfying* $\{T, H\} = 1$ *on* $M \setminus \mathsf{Crit}(H, \Phi)$ *such that*

$$\lim_{r \to \infty} \frac{1}{2} \int_0^\infty dt \left[\big(\chi_1(\Phi/r) \circ \varphi_{-t}\big)(m) - \big(\chi_1(\Phi/r) \circ \varphi_t\big)(m)\right] = T(m) \tag{Formula}$$

for each $m \in M \setminus \mathsf{Crit}(H, \Phi)$.

Remark 1:

Let $\partial_j H := \{\Phi_j, H\}$ for each j. Then, the vector $\nabla H := (\partial_1 H, \ldots, \partial_d H)$ is an abstract velocity observable for the pair (H, Φ), and

$$T = \frac{\Phi \cdot \nabla H}{(\nabla H)^2}\,.$$

Remark 1:

Let $\partial_j H := \{\Phi_j, H\}$ for each j. Then, the vector $\nabla H := (\partial_1 H, \ldots, \partial_d H)$ is an abstract velocity observable for the pair (H, Φ), and

$$T = \frac{\Phi \cdot \nabla H}{(\nabla H)^2}\,.$$

Due to the Assumption, we have once more

$$\{T, H\} = \left\{\frac{\Phi \cdot \nabla H}{(\nabla H)^2}, H\right\} = \sum_j \{\Phi_j, H\} \frac{(\partial_j H)}{(\nabla H)^2} = 1.$$

Remark 2:

We must avoid the case "$\nabla \mathsf{H} = 0$", which leads to non-definiteness of the observable T.

Remark 2:

We must avoid the case "$\nabla \mathsf{H} = 0$", which leads to non-definiteness of the observable T.

This suggests the definition:

Definition 4.2. $\mathsf{Crit}(\mathsf{H}, \Phi) := (\nabla \mathsf{H})^{-1}(\{0\}) \subset \mathsf{M}$.

Remark 2:

We must avoid the case "$\nabla \mathsf{H} = 0$", which leads to non-definiteness of the observable T.

This suggests the definition:

Definition 4.2. $\mathsf{Crit}(\mathsf{H}, \Phi) := (\nabla \mathsf{H})^{-1}(\{0\}) \subset \mathsf{M}$.

- $\mathsf{Crit}(\mathsf{H}, \Phi)$ is closed in M.

Remark 2:

We must avoid the case "$\nabla H = 0$", which leads to non-definiteness of the observable T.

This suggests the definition:

Definition 4.2. $\mathrm{Crit}(H, \Phi) := (\nabla H)^{-1}(\{0\}) \subset M$.

- $\mathrm{Crit}(H, \Phi)$ is closed in M.
- The usual critical set $\mathrm{Crit}(H)$ is contained in $\mathrm{Crit}(H, \Phi)$, *i.e.*

$$\mathrm{Crit}(H) \equiv \left\{ m \in M \mid X_H(m) = 0 \right\} \subset \mathrm{Crit}(H, \Phi).$$

Remark 2:

We must avoid the case "$\nabla H = 0$", which leads to non-definiteness of the observable T.

This suggests the definition:

Definition 4.2. $\mathsf{Crit}(H, \Phi) := (\nabla H)^{-1}(\{0\}) \subset M$.

- $\mathsf{Crit}(H, \Phi)$ is closed in M.
- The usual critical set $\mathsf{Crit}(H)$ is contained in $\mathsf{Crit}(H, \Phi)$, *i.e.*

$$\mathsf{Crit}(H) \equiv \left\{m \in M \mid X_H(m) = 0\right\} \subset \mathsf{Crit}(H, \Phi).$$

- Each orbit $\{\varphi_t(m)\}_{t \in \mathbb{R}}$ either stays in $\mathsf{Crit}(H, \Phi)$ if $m \in \mathsf{Crit}(H, \Phi)$, or stays outside $\mathsf{Crit}(H, \Phi)$ and is not periodic if $m \notin \mathsf{Crit}(H, \Phi)$.

Remark 3 (application):

Consider a scattering pair $(\mathsf{H}, \mathsf{H} + V)$ with scattering map S.

Remark 3 (application):

Consider a scattering pair $(\mathsf{H}, \mathsf{H} + V)$ with scattering map S.

The classical time delay $\tau(\mathfrak{m})$ for the initial condition $\mathfrak{m} \in \mathsf{M} \setminus \mathsf{Crit}(;\Phi)$ defined in terms of the balls B_r can be expressed as follows:

$\tau(\mathfrak{m})$ is equal to the l.h.s. of (Formula) minus the same quantity with $\mathfrak{m}$ replaced by $\mathsf{S}(\mathfrak{m})$.

Remark 3 (application):

Consider a scattering pair $(\mathsf{H}, \mathsf{H} + V)$ with scattering map S.

The classical time delay $\tau(m)$ for the initial condition $m \in \mathsf{M} \setminus \mathsf{Crit}(;\Phi)$ defined in terms of the balls B_r can be expressed as follows:

$\tau(m)$ is equal to the l.h.s. of (Formula) minus the same quantity with m replaced by $\mathsf{S}(m)$.

Thus

$$\tau(m) = (\mathsf{T} - \mathsf{T} \circ \mathsf{S})(m).$$

Remark 3 (application):

Consider a scattering pair $(\mathsf{H}, \mathsf{H}+V)$ with scattering map S.

The classical time delay $\tau(m)$ for the initial condition $m \in \mathsf{M} \setminus \mathsf{Crit}(;\Phi)$ defined in terms of the balls B_r can be expressed as follows:

$\tau(m)$ is equal to the l.h.s. of (Formula) minus the same quantity with m replaced by $S(m)$.

Thus

$$\tau(m) = (\mathsf{T} - \mathsf{T} \circ S)(m).$$

(Classical Eisenbud-Wigner Formula supporting Buslaev/Pushnitski...)

Remark 3 (application):

Consider a scattering pair $(\mathsf{H}, \mathsf{H}+V)$ with scattering map S.

The classical time delay $\tau(m)$ for the initial condition $m \in \mathsf{M} \setminus \mathsf{Crit}(;\Phi)$ defined in terms of the balls B_r can be expressed as follows:

$\tau(m)$ is equal to the l.h.s. of (Formula) minus the same quantity with m replaced by $S(m)$.

Thus

$$\tau(m) = (\mathsf{T} - \mathsf{T} \circ S)(m).$$

(Classical Eisenbud-Wigner Formula supporting Buslaev/Pushnitski...)

Furthermore, one can check that

$$\tau(m) = (\tau \circ \varphi_t)(m) \quad \text{for each } t \in \mathbb{R},$$

so classical time delay is a first integral for the free motion.

5 Examples

The framework covers many examples:

5 Examples

The framework covers many examples:

- Stark Hamiltonians,

5 Examples

The framework covers many examples:

- Stark Hamiltonians,
- Homogeneous Hamiltonians,

5 Examples

The framework covers many examples:

- Stark Hamiltonians,
- Homogeneous Hamiltonians,
- Kinetic Hamiltonians (we have seen them...),

5 Examples

The framework covers many examples:

- Stark Hamiltonians,
- Homogeneous Hamiltonians,
- Kinetic Hamiltonians (we have seen them...),
- Repulsive harmonic potential,

5 Examples

The framework covers many examples:

- Stark Hamiltonians,
- Homogeneous Hamiltonians,
- Kinetic Hamiltonians (we have seen them...),
- Repulsive harmonic potential,
- Simple pendulum,

5 Examples

The framework covers many examples:

- Stark Hamiltonians,
- Homogeneous Hamiltonians,
- Kinetic Hamiltonians (we have seen them...),
- Repulsive harmonic potential,
- Simple pendulum,
- Central force systems,

- Poincaré ball model,

- Poincaré ball model,
- Covering manifolds,

- Poincaré ball model,
- Covering manifolds,
- Wave (Klein-Gordon) equation,

- Poincaré ball model,
- Covering manifolds,
- Wave (Klein-Gordon) equation,
- Nonlinear Schrödinger equation,

- Poincaré ball model,
- Covering manifolds,
- Wave (Klein-Gordon) equation,
- Nonlinear Schrödinger equation,
- Korteweg-de Vries equation,

- Poincaré ball model,
- Covering manifolds,
- Wave (Klein-Gordon) equation,
- Nonlinear Schrödinger equation,
- Korteweg-de Vries equation,
- Quantum Hamiltonians defined via expectation values.

Example 5.1 (Poincaré ball model)**.** *Put on* $B_1 \subset \mathbb{R}^n$ *the Riemannian metric*

$$g_q(X_q, Y_q) := \frac{4}{(1 - |q|^2)^2}(X_q \cdot Y_q), \quad q \in B_1,\ X_q, Y_q \in T_q B_1 \simeq \mathbb{R}^n.$$

Example 5.1 (Poincaré ball model). *Put on* $B_1 \subset \mathbb{R}^n$ *the Riemannian metric*

$$g_q(X_q, Y_q) := \frac{4}{(1-|q|^2)^2}(X_q \cdot Y_q), \quad q \in B_1,\ X_q, Y_q \in T_q B_1 \simeq \mathbb{R}^n.$$

Consider on $M := T^*B_1 \simeq \left\{(q,p) \in B_1 \times \mathbb{R}^n\right\}$, *with symplectic form* $\omega := \sum_j dq^j \wedge dp_j$, *the kinetic energy Hamiltoninan*

$$H : M \to \mathbb{R}, \quad (q,p) \mapsto \tfrac{1}{2}\sum_{j,k} g^{jk}(q) p_j p_k = \frac{1}{8}|p|^2\left(1-|q|^2\right)^2.$$

Example 5.2 (Poincaré ball model, continued)**.**

Then:

Example 5.2 (Poincaré ball model, continued)**.**

Then:

- X_H *is complete, since the integral curves of* X_H *correspond to the geodesics curves of* (B_1, g)*, and* (B_1, g) *is geodesically complete.*

Example 5.2 (Poincaré ball model, continued)**.**

Then:

- X_H *is complete, since the integral curves of* X_H *correspond to the geodesics curves of* (B_1, g)*, and* (B_1, g) *is geodesically complete.*

- *The observable*

  $$\Phi : M \to \mathbb{R}, \quad (q, p) \mapsto e^{-1/H(q,p)} \tanh^{-1}\left(\frac{(p \cdot q)(1 - |q|^2)}{\sqrt{2H(q,p)}(1 + |q|^2)} \right).$$

  satisfies the Assumption, since

  $$\big\{\{\Phi, H\}, H\big\} = \big\{ e^{-1/H} \sqrt{2H}, H\big\} = 0.$$

Example 5.2 (Poincaré ball model, continued).

Then:

- X_H *is complete, since the integral curves of* X_H *correspond to the geodesics curves of* (B_1, g)*, and* (B_1, g) *is geodesically complete.*
- *The observable*

$$\Phi : M \to \mathbb{R}, \quad (q, p) \mapsto e^{-1/H(q,p)} \tanh^{-1}\left(\frac{(p \cdot q)(1 - |q|^2)}{\sqrt{2H(q,p)}(1 + |q|^2)}\right).$$

  satisfies the Assumption, since

$$\big\{\{\Phi, H\}, H\big\} = \big\{\, e^{-1/H} \sqrt{2H}, H\big\} = 0.$$

- $\mathrm{Crit}(H) = \mathrm{Crit}(H, \Phi) = B_1 \times \{0\}$.

Example 5.3 (NLS)**.** *Follow [Kuksin93] for the Hamiltonian formulation:*

Example 5.3 (NLS)**.** *Follow [Kuksin93] for the Hamiltonian formulation:*

Real symplectic vector space $\mathsf{M} := \mathsf{L}^2(\mathbb{R}) \oplus \mathsf{L}^2(\mathbb{R})$ *with* 2*-form*

$$\omega\big((p,q),(\widetilde{p},\widetilde{q})\big) := \big\langle (q,-p),(\widetilde{p},\widetilde{q})\big\rangle.$$

Example 5.3 (NLS). *Follow [Kuksin93] for the Hamiltonian formulation:*

Real symplectic vector space $M := L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ *with* 2*-form*

$$\omega\big((p,q),(\widetilde{p},\widetilde{q})\big) := \big\langle (q,-p),(\widetilde{p},\widetilde{q})\big\rangle.$$

Given $V, F \in C^\infty(\mathbb{R};\mathbb{R})$*, there exists* $\mathcal{O}_1 \subset \mathcal{H}^1(\mathbb{R}) \oplus \mathcal{H}^1(\mathbb{R})$ *such that*

$$H : \mathcal{O}_1 \to \mathbb{R}, \quad (p,q) \mapsto \frac{1}{2}\int_{\mathbb{R}} dx \left\{(\partial_x p)^2 + (\partial_x q)^2 + V\cdot(p^2+q^2) + F(p^2+q^2)\right\},$$

is well-defined and (Fréchet) C^∞.

Example 5.3 (NLS). *Follow [Kuksin93] for the Hamiltonian formulation:*

Real symplectic vector space $M := L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ *with* 2*-form*

$$\omega\big((p,q),(\tilde{p},\tilde{q})\big) := \big\langle (q,-p),(\tilde{p},\tilde{q})\big\rangle.$$

Given $V, F \in C^\infty(\mathbb{R};\mathbb{R})$, *there exists* $\mathcal{O}_1 \subset \mathcal{H}^1(\mathbb{R}) \oplus \mathcal{H}^1(\mathbb{R})$ *such that*

$$H : \mathcal{O}_1 \to \mathbb{R}, \quad (p,q) \mapsto \frac{1}{2}\int_{\mathbb{R}} dx \left\{(\partial_x p)^2 + (\partial_x q)^2 + V\cdot(p^2+q^2) + F(p^2+q^2)\right\},$$

is well-defined and (Fréchet) C^∞.

The corresponding equation of motion is the NLS equation

$$\frac{d}{dt}u = i\big(-\partial_x^2 u + Vu + uF'(|u|^2)\big), \quad u := p + iq.$$

Example 5.4 (NLS, continued)**.**

Then:

Example 5.4 (NLS, continued)**.**

Then:

- *The completeness of* X_H *depends on the nonlinearity term and is equivalent to the global well-posedness of the NLS ([Bourgain99], [Sulem/sulem99],* etc.*).*

Example 5.4 (NLS, continued)**.**

Then:

- *The completeness of* X_H *depends on the nonlinearity term and is equivalent to the global well-posedness of the NLS ([Bourgain99], [Sulem/sulem99],* etc.*).*

- *When* F *is arbitrary and* $V = \mathrm{Const.}$, *the function* $\Phi(p,q) := \frac{1}{2}\int_{\mathbb{R}} dx \, \mathrm{id}_{\mathbb{R}}(q^2 + p^2)$ *satisfies the Assumption, since* $\{\Phi, H\}$ *is equal to a first integral of the motion.*

Example 5.4 (NLS, continued)**.**

Then:

- *The completeness of* X_{H} *depends on the nonlinearity term and is equivalent to the global well-posedness of the NLS ([Bourgain99], [Sulem/sulem99],* etc.*).*

- *When* F *is arbitrary and* $\mathsf{V} = \mathrm{Const.}$*, the function* $\Phi(\mathsf{p},\mathsf{q}) := \frac{1}{2}\int_{\mathbb{R}} \mathrm{d}\mathsf{x}\, \mathrm{id}_{\mathbb{R}}(\mathsf{q}^2 + \mathsf{p}^2)$ *satisfies the Assumption, since* $\{\Phi, \mathsf{H}\}$ *is equal to a first integral of the motion.*

- $\mathsf{Crit}(\mathsf{H}) \subsetneq \mathsf{Crit}(\mathsf{H}, \Phi)$.

6 From classical to quantum

6 From classical to quantum

In the quantum case:

- H is a selfadjoint operator in a Hilbert space $\mathcal{H}$,

6 From classical to quantum

In the quantum case:

- H is a selfadjoint operator in a Hilbert space $\mathcal{H}$,
- $\Phi \equiv (\Phi_1, \ldots, \Phi_d)$ is a family of mutually commuting selfadjoint operators in $\mathcal{H}$ satisfying a suitable version of the commutation relation $\big[[\Phi_j, H], H\big] = 0$,

6 From classical to quantum

In the quantum case:

- H is a selfadjoint operator in a Hilbert space $\mathcal{H}$,
- $\Phi \equiv (\Phi_1, \ldots, \Phi_d)$ is a family of mutually commuting selfadjoint operators in $\mathcal{H}$ satisfying a suitable version of the commutation relation $\big[[\Phi_j, H], H\big] = 0$,
- T is a time operator for H, *i.e.* a symmetric operator satisfying the canonical commutation relation $[T, H] = i$.

6 From classical to quantum

In the quantum case:

- H is a selfadjoint operator in a Hilbert space $\mathcal{H}$,
- $\Phi \equiv (\Phi_1, \dots, \Phi_d)$ is a family of mutually commuting selfadjoint operators in $\mathcal{H}$ satisfying a suitable version of the commutation relation $\big[[\Phi_j, H], H\big] = 0$,
- T is a time operator for H, *i.e.* a symmetric operator satisfying the canonical commutation relation $[T, H] = i$.
- The confinement (resp. the non-periodicity) of the classical orbits $\{\varphi_t(m)\}_{t\in\mathbb{R}}$, $m \in M$, correspond to the affiliation of the quantum orbits $\{e^{itH}\psi\}_{t\in\mathbb{R}}$, $\psi \in \mathcal{H}$, to the singular (resp. absolutely continuous) subspace of $\mathcal{H}$.

6 From classical to quantum

In the quantum case:

- H is a selfadjoint operator in a Hilbert space $\mathcal{H}$,
- $\Phi \equiv (\Phi_1, \ldots, \Phi_d)$ is a family of mutually commuting selfadjoint operators in $\mathcal{H}$ satisfying a suitable version of the commutation relation $\big[[\Phi_j, H], H\big] = 0$,
- T is a time operator for H, *i.e.* a symmetric operator satisfying the canonical commutation relation $[T, H] = i$.
- The confinement (resp. the non-periodicity) of the classical orbits $\{\varphi_t(m)\}_{t\in\mathbb{R}}$, $m \in M$, correspond to the affiliation of the quantum orbits $\{e^{itH}\psi\}_{t\in\mathbb{R}}$, $\psi \in \mathcal{H}$, to the singular (resp. absolutely continuous) subspace of $\mathcal{H}$.
- First integrals (as τ) correspond to decomposable operators.

7 Some references

• A. Pushnitski and V. Buslaev. The scattering matrix and associated formulas in hamiltonian mechanics. *Comm. Math. Phys.*, 2010.

• A. Gournay and R. Tiedra de Aldecoa. A formula relating localisation observables to the variation of energy in Hamiltonian dynamics. `In preparation`.

• S. Richard and R. Tiedra de Aldecoa. A new formula relating localisation operators to time operators. `Submitted`.

• R. Tiedra de Aldecoa. Time delay for dispersive systems in quantum scattering theory. *Rev. Math. Phys.*, 2009.