Resolvent expansions and continuity of the scattering matrix at embedded thresholds: the case of quantum waveguides

Rafael Tiedra
(Catholic University of Chile)

Prague, January 2015

Joint work with S. Richard (University of Nagoya)

Contents

1 General setup

- $\mathcal{H}$, Hilbert space with norm $\|\cdot\|$ and scalar product $\langle\cdot,\cdot\rangle$
- $\mathscr{B}(\mathcal{H})$, bounded linear operators on $\mathcal{H}$
- H, self-adjoint operator in $\mathcal{H}$ with spectrum $\sigma(H)$
- $\mathbb{C}_\pm := \{z \in \mathbb{C} \mid \pm\,\mathsf{Im}(z) > 0\}$

Basic motivation: For $z \in \mathbb{C}_\pm$, determine the behaviour of the resolvent $R(z) := (H - z)^{-1}$ as $z \to z_0 \in \sigma(H)$.

(useful for spectral theory, scattering theory, propagation estimates, . . .)

If $v = v^* \in \mathscr{B}(\mathcal{H})$ and $u = u^* = u^{-1} \in \mathscr{B}(\mathcal{H})$ are such that

$$H = H_0 + v\,u\,v,$$

then the resolvent equation reads

$$uvR(z)vu = u - \underbrace{\big(u - vR_0(z)v\big)^{-1}}_{= A(z)^{-1} \text{ later}} \quad \text{with} \quad R_0(z) := (H_0 - z)^{-1}.$$

Example. *If* $H - H_0 = V$ *with* $V \in \mathrm{L}^\infty(\mathbb{R}^d;\mathbb{R})$, *then* $v(x) := |V(x)|^{1/2}$ *and*

$$u(x) := \begin{cases} +1 & \textit{if } V(x) \geq 0 \\ -1 & \textit{if } V(x) < 0. \end{cases}$$

2 Asymptotic expansion

Proposition. *Let* $O \subset \mathbb{C}$ *with* 0 *as accumulation point, let* $A(z) = A_0 + zA_1(z)$ *with* $A_0 \in \mathscr{B}(\mathcal{H})$ *and* $\|A_1(z)\| \leq \text{Const.}$ *for all* $z \in O$*, and let* $S = S^2 \in \mathscr{B}(\mathcal{H})$ *be such that*

(i) $A_0 + S$ *is boundedly invertible* *and* *(ii)* $S(A_0 + S)^{-1}S = S$.

Then, for $|z|$ *small enough the operator* $B(z) : S\mathcal{H} \to S\mathcal{H}$

$$B(z) := \frac{1}{z}\Big(S - S\big(A(z) + S\big)^{-1}S\Big) \equiv S(A_0{+}S)^{-1}\sum_{j\geq 0}(-z)^j\big\{A_1(z)(A_0{+}S)^{-1}\big\}^{j+1}S$$

is uniformly bounded as $z \to 0$*. Also,* $A(z)$ *is boundedly invertible in* $\mathcal{H}$ *if and only if* $B(z)$ *is boundedly invertible in* $S\mathcal{H}$*, in which case*

$$A(z)^{-1} = \big(A(z) + S\big)^{-1} + \frac{1}{z}\big(A(z) + S\big)^{-1}SB(z)^{-1}S\big(A(z) + S\big)^{-1}.$$

- The original version of this proposition is due to [Jensen-Nenciu 2001/2004] (see also [Erdoğan-Schlag 2004]).
- In the previous works, one either has that $A_0 = A_0^*$ or that S is a Riesz projection (a projection $S = S^2$ given in terms of a contour integral of the resolvent of a closed operator).

Riesz projection

There are two natural choices for S, a Riesz projection $S = S_r$ or an orthogonal projection $S = S_o$. We start with the Riesz projection.

Assumption A. 0 *is an isolated point in* $\sigma(A_0)$

Let S_r be the Riesz projection associated with $0 \in \sigma(A_0)$. Then,

$$A_0 S_r = S_r A_0 = S_r A_0 S_r \qquad \text{and} \qquad A_0 + S_r \text{ is boundedly invertible.}$$

Thus, the hypothesis (i) of the proposition is verified.

A sufficient condition for the hypothesis (ii) of the proposition is $A_0 S_r = 0$ (which is true for example if $A_0 = A_0^*$), because

$$\begin{aligned} S_r(A_0 + S_r)^{-1} S_r &= (A_0 + S_r) S_r (A_0 + S_r)^{-1} S_r \\ &= S_r (A_0 + S_r)(A_0 + S_r)^{-1} S_r \\ &= S_r \end{aligned}$$

(in general $A_0 S_r$ is only quasi-nilpotent; that is, $\sigma(A_0 S_r) = \{0\}$)

Assumption B. $\mathrm{Im}(A_0) \geq 0$

Assumption C. *$S_r A_0 S_r$ is a trass-class operator*

Lemma. *If Assumptions A,B,C are verified, then $A_0 S_r = 0$.*

Proof. The operator $J := S_r A_0 S_r$ in $S_r \mathcal{H}$ satisfies

$$\mathrm{Im} \left\langle S_r \varphi, J S_r \varphi \right\rangle = \mathrm{Im} \left\langle S_r \varphi, S_r A_0 S_r S_r \varphi \right\rangle = \mathrm{Im} \left\langle S_r \varphi, A_0 S_r \varphi \right\rangle \geq 0.$$

Since J is quasi-nilpotent and trace-class, it follows

$$\begin{aligned} 0 = \mathrm{Tr}(J) = \mathrm{Tr}\big(\mathrm{Re}(J)\big) + i \underbrace{\mathrm{Tr}\big(\mathrm{Im}(J)\big)}_{\geq 0} &\implies \mathrm{Im}(J) = 0 \\ &\implies J = J^* \\ &\implies J = 0. \end{aligned}$$

□

Thus, the hypothesis (ii) of the proposition is verified.

Orthogonal projection

Assumption B. $\mathrm{Im}(A_0) \geq 0$

Let S_o be the orthogonal projection on

$$\ker(A_0) \equiv \ker\big(\mathrm{Re}(A_0)\big) \cap \ker\big(\mathrm{Im}(A_0)\big) \equiv \ker(A_0^*).$$

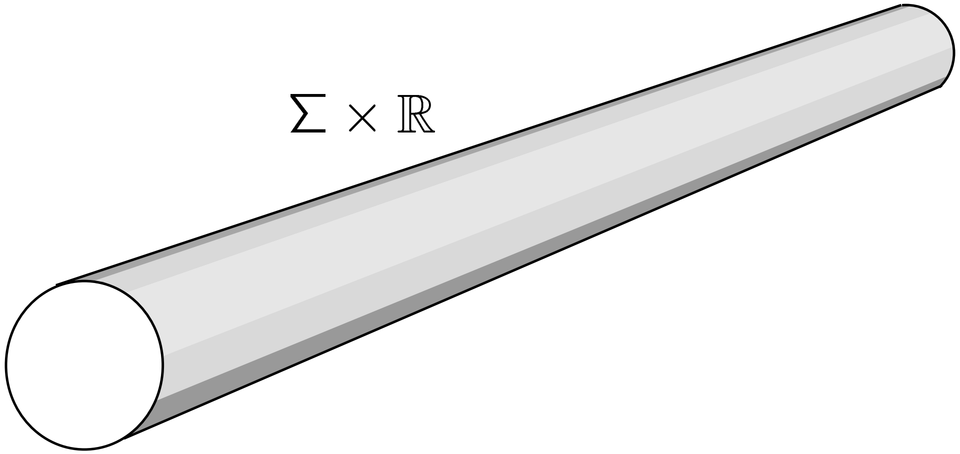
Then, $A_0 S_o = 0$, and thus the hypotheses (i) and (ii) of the proposition are verified if $A_0 + S_o$ is boundedly invertible.

Two cases in which $A_0 + S_o$ is boundedly invertible:

Lemma. *If Assumptions A,B,C are verified, then* $A_0 + S_o$ *is boundedly invertible if and only if* $S_r = S_r^* = S_o$.

Lemma. *If Assumption B is verified and if* A_0 *is a finite-rank operator or* $A_0 = U + K$ *with* U *unitary and* K *compact, then* $A_0 + S_o$ *is boundedly invertible.*

3 Application to quantum waveguides



- Σ, bounded open connected set in $\mathbb{R}^{d-1}$, $d \geq 2$,
- $\Omega := \Sigma \times \mathbb{R}$
- $\mathcal{H} := \mathsf{L}^2(\Omega) \simeq \mathsf{L}^2(\Sigma) \otimes \mathsf{L}^2(\mathbb{R})$

Free Hamiltonian and perturbed Hamiltonian

$$H_0 := -\triangle_{\mathrm{D}}^{\Sigma} \otimes 1 + 1 \otimes (-\triangle^{\mathbb{R}}) \qquad \text{and} \qquad H := H_0 + V,$$

with $-\triangle_{\mathrm{D}}^{\Sigma}$ the Dirichlet Laplacian on Σ and $V \in \mathsf{L}^\infty(\Omega; \mathbb{R})$ of compact support.

The Dirichlet Laplacian $-\triangle_{\mathrm{D}}^{\Sigma}$ has purely discrete spectrum

$$\tau := \{\lambda_n\}_{n \geq 1}$$

consisting in eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots$ repeated according to multiplicity (these are the embedded thresholds). $\mathcal{P}_n$ is the orthogonal projection associated with λ_n.

Known facts (see for instance [T. 2006]):

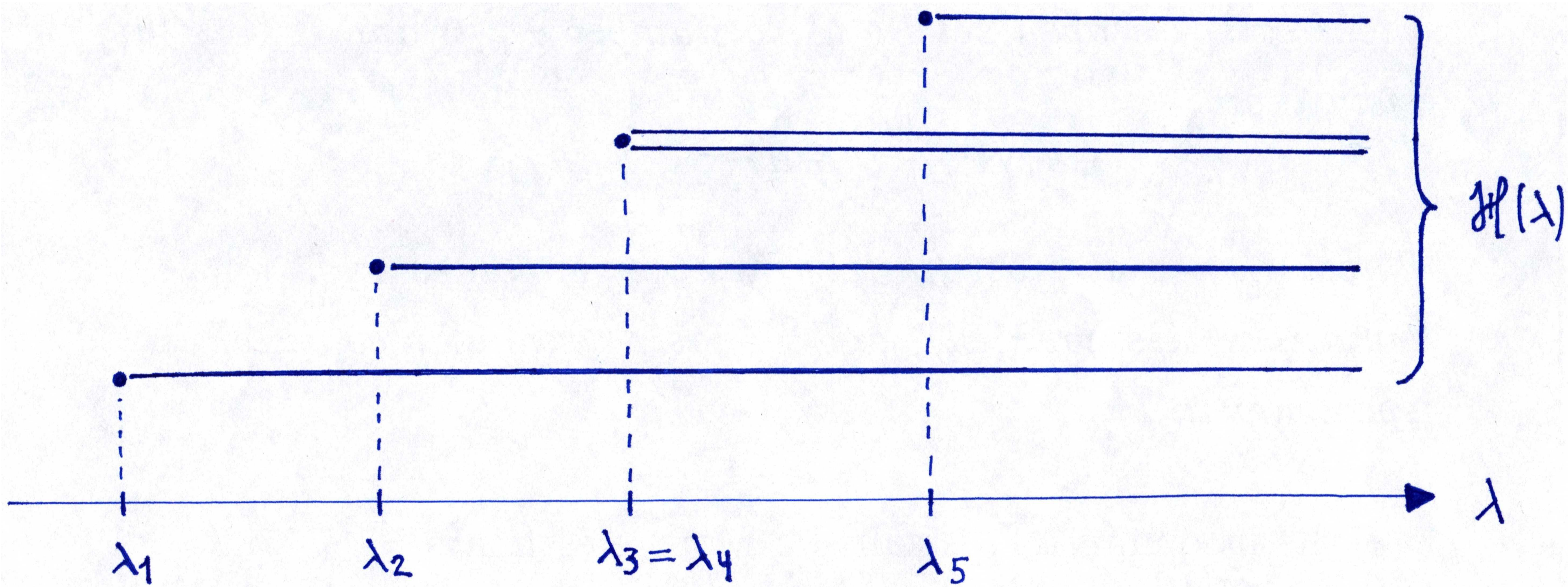
- $\sigma(H) = \sigma_{\mathrm{ess}}(H) = \sigma_{\mathrm{ac}}(H) = [\lambda_1, \infty)$
- $\sigma_{\mathrm{p}}(H)$ can accumulate at points of τ only
- the wave operators $W_\pm := \mathrm{s}\text{-}\lim_{t\to\pm\infty} \mathrm{e}^{itH}\,\mathrm{e}^{-itH_0}$ exist and are complete
- the scattering operator $S := W_+^* W_-$ is unitary

S is decomposable in the spectral representation of H_0 as follows.
For $\lambda \geq \lambda_1$, set

$$\mathbb{N}(\lambda) := \{n \geq 1 \mid \lambda_n \leq \lambda\}$$

and

$$\mathcal{H}(\lambda) := \bigoplus_{n \in \mathbb{N}(\lambda)} \{\mathcal{P}_n \mathsf{L}^2(\Sigma) \oplus \mathcal{P}_n \mathsf{L}^2(\Sigma)\}.$$



There is a unitary operator $\mathscr{F}_0 : \mathcal{H} \to \int_{[\lambda_1,\infty)}^{\oplus} \mathcal{H}(\lambda)\,\mathrm{d}\lambda$ such that

$$\mathscr{F}_0 H_0 \mathscr{F}_0^* = \int_{[\lambda_1,\infty)}^{\oplus} \lambda\,\mathrm{d}\lambda \qquad \text{and} \qquad \mathscr{F}_0\, S \mathscr{F}_0^* = \int_{[\lambda_1,\infty)}^{\oplus} S(\lambda)\,\mathrm{d}\lambda,$$

with $S(\lambda)$ unitary in $\mathcal{H}(\lambda)$ for a.e. $\lambda \geq \lambda_1$, and with

$$[\lambda_1,\infty) \setminus \big\{\tau \cup \sigma_{\mathrm{p}}(H)\big\} \mapsto S(\lambda) \in \mathscr{B}\big(\mathcal{H}(\lambda)\big)$$

of class C^∞.

(it remains to determine the behavior of $S(\lambda)$ as $\lambda \to \lambda_0 \in \tau \cup \sigma_{\mathrm{p}}(H) \ldots$)

For a.e. $\lambda \geq \lambda_1$, let $S(\lambda) \equiv \{S_{nn'}(\lambda)\}_{n,n' \in \mathbb{N}(\lambda)}$ with

$$S_{nn'}(\lambda) : \mathcal{P}_{n'} \mathsf{L}^2(\Sigma) \to \mathcal{P}_n \mathsf{L}^2(\Sigma).$$

The behaviour of $S_{nn'}(\lambda)$ as $\lambda \to \lambda_0 \in \tau$ is the following:

Theorem ([Richard, T. 2014]). *Let $\lambda_m \in \tau$ and $n, n' \geq 1$. Then,*

(a) if $\lambda_n, \lambda_{n'} < \lambda_m$, the map $\lambda \mapsto S_{nn'}(\lambda)$ is continuous in a neighbourhood of λ_m,

(b) if $\lambda_n, \lambda_{n'} \leq \lambda_m$, the limit $\lim_{\varepsilon \searrow 0} S_{nn'}(\lambda_m + \varepsilon)$ exists.

- One cannot ask for more continuity in (b), since a channel could open at the energy λ_m.
- The case $\lambda \to \lambda_0 \in \sigma_{\mathrm{p}}(H)$ is easier to treat.

Idea of the proof. Use a stationary representation

$$S(\lambda) = 1_{\mathcal{H}(\lambda)} - 2\pi i\, \mathscr{F}_0(\lambda)\, v\big(u + vR_0(\lambda + i\,0)v\big)^{-1} v\, \mathscr{F}_0(\lambda)^*$$

with $\mathscr{F}_0(\lambda)\varphi := (\mathscr{F}_0\varphi)(\lambda)$, and then apply iteratively Proposition 2 to get an asymptotic expansion for $\big(u + vR_0(\lambda_m + \varepsilon)v\big)^{-1}$ for suitable small $\varepsilon \in \mathbb{C}$.

An asymptotic expansion for $\mathscr{F}_0(\lambda_m + \varepsilon)$ is also necessary. $\square$

- In the proof, the iterations stop because

$$uvR(\lambda_m+\varepsilon)\,vu = u - \big(u + vR_0(\lambda_m+\varepsilon)v\big)^{-1}$$

and for suitable ε (such as $\varepsilon = \pm i\,\delta$, $\delta > 0$)

$$\big\|\varepsilon\, R(\lambda_m+\varepsilon)\big\| \le 1 \implies \limsup_{\varepsilon\to 0} \big\|\varepsilon\,\big(u+vR_0(\lambda_m+\varepsilon)v\big)^{-1}\big\| < \infty.$$

- Another consequence of the asymptotic expansion for $\big(u + vR_0(\lambda_m+\varepsilon)v\big)^{-1}$ is the absence of accumulation of eigenvalues of H at the points of τ.

Gracias !

4 References

• A. Jensen and G. Nenciu. A unified approach to resolvent expansions at thresholds. *Rev. Math. Phys.*, 2001/2004

• S. Richard and R. Tiedra. Resolvent expansions and continuity of the scattering matrix at embedded thresholds: the case of quantum waveguides. to appear in Bull. Soc. Math. France

• R. Tiedra. Time delay and short-range scattering in quantum waveguides. *Ann. Henri Poincaré*, 2006