The absolute continuous spectrum of skew products of compact Lie groups

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Contents

1 Unitary operators

A unitary operator U in a Hilbert space $\mathcal{H}$ is a surjective isometry:

$$U^*U = UU^* = 1.$$

It admits exactly one complex spectral family

$$E^U : \mathbb{C} \to \{\text{orthogonal projections on } \mathcal{H}\}$$

with support

$$\sigma(U) \subset \mathbb{S}^1 \equiv \{z \in \mathbb{C} \mid |z| = 1\}$$

such that

$$U = \int_{\mathbb{C}} z \, \mathrm{d}E^U(z).$$

The spectrum $\sigma(U)$ of U decomposes as

$$\sigma(U) = \sigma_{\mathrm{p}}(U) \cup \sigma_{\mathrm{sc}}(U) \cup \sigma_{\mathrm{ac}}(U),$$

with

$$\begin{aligned} \sigma_{\mathrm{p}}(U) &:= \text{ pure point spectrum of } U, \\ \sigma_{\mathrm{sc}}(U) &:= \text{ singular continuous spectrum of } U, \\ \sigma_{\mathrm{ac}}(U) &:= \text{ absolutely continuous spectrum of } U. \end{aligned}$$

The sets $\sigma_{\mathrm{p}}(U)$, $\sigma_{\mathrm{sc}}(U)$, $\sigma_{\mathrm{ac}}(U)$ are closed and (in general) not mutually disjoint.

Example (1-parameter groups of unitary operators). *If H is a self-adjoint operator in a Hilbert space $\mathcal{H}$, then*

$$U_t := \mathrm{e}^{-itH}, \quad t \in \mathbb{R},$$

defines a strongly continuous 1*-parameter group of unitary operators.*

Example (Koopman operator). *If $T : X \to X$ is an automorphism of a probability space (X, μ), then the Koopman operator*

$$U_T : \mathsf{L}^2(X, \mu) \to \mathsf{L}^2(X, \mu), \quad \varphi \mapsto \varphi \circ T,$$

is a unitary operator.

Ergodicity, weak mixing and strong mixing of an automorphism $T : X \to X$ are expressible in terms of spectral properties of the Koopman operator U_T :

- T is ergodic if and only if 1 is a simple eigenvalue of U_T.
- T is weakly mixing if and only if U_T has purely continuous spectrum in $\{\mathbb{C} \cdot 1\}^{\perp}$.
- T is strongly mixing if and only if

$$\lim_{n\to\infty} \langle \varphi, U_T^n \varphi \rangle = 0 \quad \text{for all } \varphi \in \{\mathbb{C} \cdot 1\}^{\perp}.$$

a.c. spectrum in $\{\mathbb{C} \cdot 1\}^{\perp}$ $\Rightarrow$ strong mixing $\Rightarrow$ weak mixing $\Rightarrow$ ergodicity

2 Commutator methods

- $\mathcal{H}$, Hilbert space with norm $\|\cdot\|$ and scalar product $\langle\cdot,\cdot\rangle$
- $\mathscr{B}(\mathcal{H})$, bounded linear operators on $\mathcal{H}$
- $\mathscr{K}(\mathcal{H})$, compact operators on $\mathcal{H}$
- U, unitary operator in $\mathcal{H}$
- A, self-adjoint operator in $\mathcal{H}$ with domain $\mathcal{D}(A) \subset \mathcal{H}$

Definition. *$S \in \mathscr{B}(\mathcal{H})$ satisfies $S \in C^k(A)$ if*

$$\mathbb{R} \ni t \mapsto \mathrm{e}^{-itA}\, S\, \mathrm{e}^{itA} \in \mathscr{B}(\mathcal{H})$$

is strongly of class C^k.

$S \in C^1(A)$ if and only if

$$\big|\langle \varphi, SA\varphi\rangle - \langle A\varphi, S\varphi\rangle\big| \le \text{Const.}\, \|\varphi\|^2 \quad \text{for all } \varphi \in \mathcal{D}(A).$$

The operator corresponding to the continuous extension of the quadratic form is denoted by $[S, A]$, and one has

$$[iS, A] = \text{s-}\frac{\mathrm{d}}{\mathrm{d}t}\, \mathrm{e}^{-itA}\, S\, \mathrm{e}^{itA}\Big|_{t=0} \in \mathscr{B}(\mathcal{H}).$$

Example. *Let P be the generator of translations in $\mathcal{H} := \mathsf{L}^2(\mathbb{R})$, let $f \in \mathsf{L}^\infty(\mathbb{R})$ be an a.c. function with $f' \in \mathsf{L}^\infty(\mathbb{R})$, and let*

$$M_f\,\varphi := f\,\varphi, \quad \varphi \in \mathcal{H},$$

the corresponding bounded multiplication operator.

One has for each $\varphi \in \mathcal{H}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{e}^{-itP}\,M_f\,\mathrm{e}^{itP}\,\varphi = \frac{\mathrm{d}}{\mathrm{d}t}\,M_{f(\,\cdot\,-t)}\,\varphi = -M_{f'(\,\cdot\,-t)}\,\varphi,$$

and thus $M_f \in C^1(P)$ with $[iM_f, P] = -M_{f'}$.

Definition. *$S \in C^{1+0}(A)$ if $S \in C^1(A)$ and*

$$\int_0^1 \frac{\mathrm{d}t}{t} \left\| \mathrm{e}^{-itA}[A,S]\,\mathrm{e}^{itA} - [A,S] \right\|_{\mathscr{B}(\mathcal{H})} < \infty.$$

We have the inclusions

$$C^2(A) \subset C^{1+0}(A) \subset C^1(A) \subset C^0(\mathcal{H}) \equiv \mathscr{B}(\mathcal{H}).$$

Example (Continued)**.** *Let* $f \in \mathsf{L}^\infty(\mathbb{R})$ *be an a.c. function with* $f' \in \mathsf{L}^\infty(\mathbb{R})$ *Dini-continuous.*

We know that $M_f \in C^1(P)$ *with* $[iM_f, P] = -M_{f'}$*. So,*

$$\begin{aligned}\int_0^1 \frac{\mathrm{d}t}{t} \left\| \mathrm{e}^{-itP}[M_f, P]\,\mathrm{e}^{itP} - [M_f, P] \right\|_{\mathscr{B}(\mathcal{H})} &= \int_0^1 \frac{\mathrm{d}t}{t} \left\| M_{f'(\,\cdot\,-t)-f'} \right\|_{\mathscr{B}(\mathcal{H})} \\ &= \int_0^1 \frac{\mathrm{d}t}{t} \left\| f'(\,\cdot\, - t) - f' \right\|_{\mathsf{L}^\infty(\mathbb{R})} \\ &< \infty\end{aligned}$$

due to the Dini-continuity of f'*, and thus* $M_f \in C^{1+0}(P)$*.*

Theorem ([Fernández/Richard/T. 2013]). *Let $U \in C^{1+0}(A)$. Assume there exist an open set $\Theta \subset \mathbb{S}^1$, a number $a > 0$ and $K \in \mathscr{K}(\mathcal{H})$ such that*

$$E^U(\Theta)\,U^*[A,U]\,E^U(\Theta) \geq a\,E^U(\Theta) + K. \qquad (\bigstar)$$

Then, U has at most finitely many eigenvalues in Θ (multiplicities counted), and U has no singular continuous spectrum in Θ.

- The inequality $(\bigstar)$ is called a Mourre estimate for U on Θ.
- The operator A is called a conjugate operator for U on Θ.
- If $K = 0$, then U is purely absolutely continuous in $\Theta \cap \sigma(U)$.

3 Skew products of compact Lie groups

- X, G, compact Lie groups with Haar measures μ_X and μ_G
- $F_1 : X \to X$, time-one map of a C^∞ measure-preserving flow $\{F_t\}_{t\in\mathbb{R}}$ on (X, μ_X)
- $\phi \in C(X; G)$, continuous function

ϕ induces a cocycle $X \times \mathbb{Z} \ni (x, n) \mapsto \phi^{(n)}(x) \in G$ over F_1 given by

$$\phi^{(n)}(x) := \begin{cases} \phi(x)(\phi \circ F_1)(x) \cdots (\phi \circ F_{n-1})(x) & \text{if } n \geq 1 \\ e_G & \text{if } n = 0 \\ \big(\phi^{(-n)} \circ F_n\big)(x)^{-1} & \text{if } n \leq -1. \end{cases}$$

The skew product

$$T_\phi : X \times G \to X \times G, \quad (x, g) \mapsto \big(F_1(x), g\,\phi(x)\big),$$

is an automorphism of the measure space $(X \times G, \mu_X \otimes \mu_G)$.

We study the continuous spectrum of the corresponding Koopman operator

$$U_\phi \psi := \psi \circ T_\phi, \quad \psi \in \mathcal{H} := \mathsf{L}^2(X \times G, \mu_X \otimes \mu_G),$$

in the spirit of [Anzai 1951], [Iwanik/Lemańzyk/Rudolph 1993], [Frączek 2000/2004], ...

The Peter-Weyl theorem induces the orthogonal decomposition

$$\mathcal{H} = \bigoplus_{\pi \in \widehat{G}} \bigoplus_{j=1}^{d_\pi} \mathcal{H}_j^{(\pi)}, \quad \mathcal{H}_j^{(\pi)} := \left\{ \sum_{k=1}^{d_\pi} \varphi_k \otimes \pi_{jk} \mid \varphi_k \in \mathsf{L}^2(X, \mu_X) \right\},$$

where

- $\widehat{G}$ is the set of all equivalence classes of finite-dimensional irreducible unitary representations (IUR) of G,
- $\pi_{jk} \in \mathsf{L}^2(G, \mu_G)$ are the matrix elements of $\pi \in \widehat{G}$.

U_ϕ is reduced by the decomposition, and $U_{\pi,j} := U_\phi\big|_{\mathcal{H}_j^{(\pi)}}$ is given by

$$U_{\pi,j} \sum_{k=1}^{d_\pi} \varphi_k \otimes \pi_{jk} = \sum_{k,\ell=1}^{d_\pi} (\varphi_k \circ F_1)(\pi_{\ell k} \circ \phi) \otimes \pi_{j\ell}\,, \quad \varphi_k \in \mathsf{L}^2(X, \mu_X).$$

→ The problem reduces to the study of the continuous spectrum of the operators $U_{\pi,j}$.

4 The conjugate operator

Let $\{V_t\}_{t\in\mathbb{R}}$ be the unitary group

$$V_t\varphi := \varphi \circ F_t, \quad \varphi \in \mathsf{L}^2(X, \mu_X),$$

and $H = -i\mathscr{L}_Y$ its self-adjoint generator (Y is the divergence-free vector field associated to $\{F_t\}_{t\in\mathbb{R}}$ and $\mathscr{L}_Y$ its Lie derivative).

The operator

$$A\sum_{k=1}^{d_\pi}\varphi_k \otimes \pi_{jk} := \sum_{k=1}^{d_\pi} a_k H\varphi_k \otimes \pi_{jk}, \quad a_k \in \mathbb{R},\ \varphi_k \in C^\infty(X),$$

is essentially self-adjoint in $\mathcal{H}_j^{(\pi)}$.

Assumption. *For each k, ℓ, the function $\pi_{k\ell} \circ \phi \in C(X;\mathbb{C})$ has a derivative $\mathscr{L}_Y(\pi_{k\ell} \circ \phi)$ which is Dini-continuous along the flow:*

$$\int_0^1 \frac{\mathrm{d}t}{t} \left\| \mathscr{L}_Y(\pi_{k\ell} \circ \phi) \circ F_t - \mathscr{L}_Y(\pi_{k\ell} \circ \phi) \right\|_{\mathsf{L}^\infty(X)} < \infty,$$

and $(a_k - a_\ell)(\pi_{\ell k} \circ \phi) \equiv 0$.

Then, one has $U_{\pi,j} \in C^{1+0}(A)$ with $[A, U_{\pi,j}] = MU_{\pi,j}$ and

$$M \sum_{k=1}^{d_\pi} \varphi_k \otimes \pi_{jk} := \sum_{k,\ell=1}^{d_\pi} M_{k\ell}\, \varphi_\ell \otimes \pi_{jk}, \quad M_{k\ell} := -i a_k \big\{ \mathscr{L}_Y(\pi \circ \phi) \cdot (\pi^* \circ \phi) \big\}_{k\ell}.$$

(M is a hermitian matrix-valued multiplication operator in $\mathcal{H}_j^{(\pi)}$)

So, one has

$$(U_{\pi,j})^*[A, U_{\pi,j}] = (U_{\pi,j})^* M U_{\pi,j},$$

and one gets a global Mourre estimate

$$(U_{\pi,j})^*[A, U_{\pi,j}] \geq \lambda_* \quad \text{if} \quad M \geq \lambda_* > 0.$$

Let's see a refinement of this idea taking into account the possible ergodicity of F_1.

The average of A along the flow generated by $U_{\pi,j}$ is a self-adjoint operator :

$$A_N\,\varphi := \frac{1}{N}\sum_{n=0}^{N-1}(U_{\pi,j})^n A\,(U_{\pi,j})^{-n}\varphi, \quad N \in \mathbb{N}_{\geq 1},\ \varphi \in \mathcal{D}(A_N) := \mathcal{D}(A).$$

One obtains $U_{\pi,j} \in C^{1+0}(A_N)$ with $[A_N, U_{\pi,j}] = M_N U_{\pi,j}$ where

$$M_N := \frac{1}{N}\sum_{n=0}^{N-1}\big(\pi \circ \phi^{(n)}\big)(M \circ F_n)\big(\pi^* \circ \phi^{(n)}\big).$$

(M_N is an average of M; we will come back to this)

With the notation

$$\lambda_{*,N} := \inf_{k\in\{1,\dots,d_\pi\},\, x\in X} \lambda_k\big(M_N(x)\big),$$

one thus obtains:

Theorem ([T. 2013])**.** *Suppose that the previous assumptions are satisfied and assume that* $\lambda_{*,N} > 0$ *for some* $N \in \mathbb{N}_{\geq 1}$. *Then,* $U_{\pi,j}$ *satisfies the global Mourre estimate*

$$(U_{\pi,j})^*[A_N, U_{\pi,j}] \geq \lambda_{*,N},$$

and $U_{\pi,j}$ *has purely absolutely continuous spectrum.*

Remark (Topological degree). *One has*

$$M_N = \cdots = D_a \frac{1}{N} \mathscr{L}_Y \big((\pi \circ \phi)^{(N)} \big) \cdot \big((\pi \circ \phi)^{(N)} \big)^*$$

with

$$D_a := -i \, \mathsf{diag}(a_1, \ldots, a_{d_\pi}).$$

Thus, if $N \gg 1$, M_N *is close to* D_a *times the (matricial) topological degree of* $\pi \circ \phi$ *.*

So, the condition $\lambda_{*,N} > 0$ *means that the topological degree of* $\pi \circ \phi$ *has nonzero determinant, in which case* U_ϕ *has purely absolutely continuous spectrum in the subspace associated to* π*.*

We can apply the theorem to various cases where the IUR of G are known.

For instance, we can treat the cases where F_1 is an ergodic translation on $X = \mathbb{T}^d$ and

- $G = \mathbb{T}^{d'}$,
- $G = \mathsf{SU}(2)$,
- $G = \mathsf{U}(2)$

$\vdots$

5 The case $\mathbb{T} \times \mathsf{SU}(2)$

Take $X = \mathbb{T}$ and $G = \mathsf{SU}(2)$, let

$$\pi^{(n)} : \mathsf{SU}(n) \to \mathsf{U}(V_n) \simeq \mathsf{U}(n+1)$$

be a $(n+1)$-dimensional IUR of $\mathsf{SU}(n)$ on the vector space V_n of homogeneous polinomials of degree n in two variables, and set

$$F_t(x) := x + t\,y \pmod{\mathbb{Z}}, \quad t \in \mathbb{R},\ x \in \mathbb{T},\ y \in \mathbb{R} \setminus \mathbb{Q}.$$

Suppose that

$$\phi(x) := h \begin{pmatrix} \mathrm{e}^{2\pi i(bx+\eta(x))} & 0 \\ 0 & \mathrm{e}^{-2\pi i(bx+\eta(x))} \end{pmatrix} h^*, \quad x \in \mathbb{T},$$

with $h \in \mathsf{SU}(2)$, $b \in \mathbb{Z} \setminus \{0\}$ and $\eta \in C^1(\mathbb{T};\mathbb{R})$ such that

$$\int_0^1 \frac{\mathrm{d}t}{t} \left\| \eta' \circ F_t - \eta' \right\|_{\mathsf{L}^\infty(\mathbb{T})} < \infty.$$

Then,

$$M_{jk} = \cdots = (\text{something frightening}) \cdot \delta_{jk},$$

but we can choose the scalars a_j so that we get

$$M_{jk} = \big(1 + (yb)^{-1}\eta'\big)(2j - n)^2\,\delta_{jk}.$$

It follows by unique ergodicity of F_1 that

$$\begin{aligned}\lim_{N\to\infty} M_N &= \left(1 + (yb)^{-1} \lim_{N\to\infty} \frac{1}{N}\sum_{m=0}^{N-1} \eta' \circ F_m\right) \begin{pmatrix} (2\cdot 0 - n)^2 & & 0 \\ & \ddots & \\ 0 & & (2\cdot n - n)^2 \end{pmatrix} \\ &= \begin{pmatrix} (2\cdot 0 - n)^2 & & 0 \\ & \ddots & \\ 0 & & (2\cdot n - n)^2 \end{pmatrix}\end{aligned}$$

uniformly on $\mathbb{T}$.

So, M_N is strictly postive if $n \in 2\mathbb{N}+1$ and $N \gg 1$, and thus

$$\lambda_{*,N} > 0 \quad \text{if} \quad N \gg 1.$$

Therefore, the theorem applies and $U_{\pi^{(n)},j}$, $j \in \{0,\dots,n\}$, has purely absolutely continuous spectrum (in fact Lebesgue spectrum).

It follows that the restriction of U_ϕ to the subspace

$$\bigoplus_{n\in 2\mathbb{N}+1} \bigoplus_{j=0}^{n} \mathcal{H}_j^{(\pi^{(n)})} \subset \mathcal{H}$$

has countable Lebesgue spectrum.

6 References

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