Toeplitz algebras and spectral results for the one-dimensional Heisenberg model

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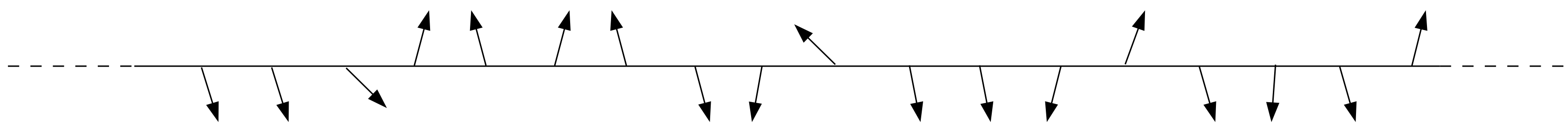
1 One-dimensional Heisenberg model



- $\mathbb{Z}$, one-dimensional lattice of spins-$\frac{1}{2}$
- $\{e^0, e^1\} := \{(0,1),(1,0)\}$, basis of the spin-$\frac{1}{2}$ Hilbert space $\mathbb{C}^2$
- $\mathbb{F}(\mathbb{Z}) := \{\alpha : \mathbb{Z} \to \{0,1\} : \text{supp}(\alpha) \text{ is finite}\}$, "spins configurations space"
- $e^\alpha \equiv \{e^{\alpha(x)}\}_{x\in\mathbb{Z}}$, $\alpha \in \mathbb{F}(\mathbb{Z})$, pure state of the spins chain
- e^{α_0}, $\alpha_0(x) := 0 \ \forall x \in \mathbb{Z}$, fundamental state with all spins down
- Hilbert space (incomplete tensorial product indexed by e^{α_0}) :

$$\mathcal{L} := \bigotimes_{x\in\mathbb{Z}}^{\alpha_0} \mathbb{C}^2_x \equiv \text{closed span}\{e^\alpha : \alpha \in \mathbb{F}(\mathbb{Z})\}$$

(Von Neumann, 1939)

One-dimensional ferromagnetic Heisenberg Hamiltonian :

$$L := -\frac{1}{2} \sum_{|x-y|=1} \left[\sigma_1^{(x)} \sigma_1^{(y)} + \sigma_2^{(x)} \sigma_2^{(y)} + \sigma_3^{(x)} \sigma_3^{(y)} - 1\right].$$

$\sigma_j^{(x)}$ acts in $\mathcal{L}$ as the identitiy on each factor $\mathbb{C}_y^2$, except on $\mathbb{C}_x^2$, where it acts as the Pauli matrix σ_j.

(Streater, 70's)

The Heisenberg Hamiltonian is unitarily equivalent to a finite difference operator $\widetilde{L}$ acting in $\ell^2[\mathbb{F}(\mathbb{Z})]$:

For $\alpha \in \mathbb{F}(\mathbb{Z})$, $x \in \operatorname{supp}(\alpha)$, $y \notin \operatorname{supp}(\alpha)$, and $\alpha_x^y \in \mathbb{F}(\mathbb{Z})$ such that $\operatorname{supp}(\alpha_x^y) = \operatorname{supp}(\alpha) \sqcup \{y\} \setminus \{x\}$, we have

$$\big(\widetilde{L}f\big)(\alpha) = -2 \sum_{|x-y|=1} \alpha(x)\,[1-\alpha(y)]\,\big[f(\alpha_x^y) - f(\alpha)\big]$$

for each $f \in \ell^2[\mathbb{F}(\mathbb{Z})]$.

Remark 1.1. *The subspace* $\mathcal{H}_N := \ell^2[\mathbb{F}_N(\mathbb{Z})]$, *where* $\mathbb{F}_N(\mathbb{Z}) := \{\alpha \in \mathbb{F}(\mathbb{Z}) : \#\operatorname{supp}(\alpha) = N\}$, *is left invariant by* $\widetilde{L}$.

Thus we study the (bounded) Heisenberg Hamiltonian in the N-magnons sector :

$$H_N := \widetilde{L} \restriction \mathcal{H}_N.$$

2 "Generalized" Toeplitz algebras

– X, discrete abelian group (not necessarily ordered)
– E, nonvoid subset of X (not necessarily the positive cone of X)
– $P^E : \ell^2(X) \to \ell^2(E)$, orthogonal projection

For $\eta \in X$, we define the unitary operator (bilateral shift) u_η in $\ell^2(X)$:

$$(u_\eta f)(\xi) := f(\xi - \eta), \quad f \in \ell^2(X), \ \xi \in X.$$

We also have the partial isometries (unilateral shift) v_η^E in $\ell^2(E)$:

$$v_\eta^E := P^E u_\eta \restriction \ell^2(E).$$

Definition 2.1. *The* C^**-algebra* $\mathcal{T}^E(X) \subset \mathscr{B}[\ell^2(E)]$ *generated by the family* $\left\{v_\eta^E\right\}_{\eta \in X}$ *is called* ***Toeplitz algebra for the group*** X ***w.r.t. the subset*** E*.*

Example 2.2. $\mathcal{T} := \mathcal{T}^{\mathbb{N}}(\mathbb{Z})$ *(*$\equiv \mathcal{T}^{\mathbb{Z}_+}(\mathbb{Z})$*)*

Example 2.3. $\mathcal{T}_m := \mathcal{T}^{(\mathbb{Z}^m_{\mathrm{lex}})_+}(\mathbb{Z}^m_{\mathrm{lex}})$ *(Murphy, 90's),*

Example 2.4. $\mathcal{T}^<(\mathbb{Z}^N) := \mathcal{T}^{\mathbb{Z}^N_<}(\mathbb{Z}^N)$, *where*
$\mathbb{Z}^N_< := \{(x_1, \dots, x_N) \in \mathbb{Z}^N : x_1 < x_2 < \cdots < x_N\}$

If $\varphi \in \ell^1(X)$, then the algebra $\mathcal{T}^E(X)$ contains the "Toeplitz operators"

$$T_\varphi^E := \sum_{\eta \in X} \varphi(\eta) v_\eta^E$$

and the "potentials"

$$V_\varphi^E := \sum_{\eta \in X} \varphi(\eta) q_\eta^E \equiv \sum_{\eta \in X} \varphi(\eta) v_\eta^E \left(v_\eta^E\right)^* .$$

– S is the collection of vectors $\{s_i^\pm\}_{i=1}^N \subset \mathbb{Z}^N$ with componants $(s_i^\pm)_j := \pm\delta_{ij}$.

Proposition 2.5. *The Hamiltonian* H_N *is unitarily equivalent to the operator* $T_\varphi^< + V_\psi^< \in \mathcal{T}^<(\mathbb{Z}^N)$*, where* $\varphi := -2\chi_S$ *and* $\psi := 2\chi_S$*.*

Sketch of the proof. Conjugate H_N by the unitary operator

$$\Phi : \mathcal{H}_N \to \ell^2(\mathbb{Z}^N_<), \quad f \mapsto f \circ \phi,$$

where $\phi : \mathbb{Z}^N_< \to \mathbb{F}_N(\mathbb{Z})$, $(x_1, \ldots, x_N) \mapsto \chi_{\{x_1,\ldots,x_N\}}$. □

Remark 2.6. *(a) Some spectral and progation properties of* H_N *can be deduced directly from the structure of* $\mathcal{T}^<(\mathbb{Z}^N)$*.*

(b) When studying the algebra $\mathcal{T}^<(\mathbb{Z}^N)$*, one studies* ***all*** *operators belonging to it (not only the Heisenberg Hamiltonian).*

3 Structure of $\mathcal{T}^{<}(\mathbb{Z}^N)$

- Group automorphism

$$\theta : \mathbb{Z}^N \to \mathbb{Z}^N, \quad (y_1, \ldots, y_n) \mapsto (y_1, y_2 - y_1, \ldots, y_N - y_{N-1})$$

- For each $\tau \in \mathbb{T}$, $\mu(\tau) : \ell^1(\mathbb{Z}^N) \to \ell^1(\mathbb{Z}^{N-1})$ is defined by

$$[\mu(\tau)\rho](z_2, \ldots, z_N) := [\mathscr{F}_1(\rho \circ \theta^{-1})](\tau, z_2, \ldots, z_N),$$

where $\mathscr{F}_j$ is the partial Fourier transform in the j^{th} variable

Lemma 3.1. *The* C^**-algebras* $\mathcal{T}^<(\mathbb{Z}^N)$ *and* $C(\mathbb{T}) \otimes \mathcal{T}^{\otimes(N-1)}$ *are isomorphic. The isomorphism sends* $T_\varphi^< + V_\psi^<$ *onto the direct integral*

$$\int_{\mathbb{T}}^{\oplus} d\tau \left(T^{N-1}_{\mu(\tau)\varphi} + V^{N-1}_{\mu(0)\psi} \right),$$

where the exponant of the Toeplitz operators refers to the subset $(\mathbb{N}^*)^{N-1}$ *of the group* $\mathbb{Z}^{N-1}$.

Outline of the proof. We use the following facts.

(A) If $\theta : X \to X'$ is a group isomorphism sending $E \subset X$ onto $E' \subset X'$, then $\mathcal{T}^E(X)$ and $\mathcal{T}^{E'}(X')$ are isomorphic.

(B) If E_j is a subset of a group X_j, $j = 1, \ldots, k$, then $\mathcal{T}^{E_1 \times \cdots \times E_k}(X_1 \times \cdots \times X_k)$ can be identified to the (spatial) tensorial product $\bigotimes_{j=1}^k \mathcal{T}^{E_j}(X_j)$. $\square$

Remark 3.2. *The occurrence of direct integrals is related to the invariance of the operators under the natural action of the group* $\mathbb{Z}$ *on* $\mathbb{Z}^N_<$.

Corollary 3.3. *For all real functions* $\varphi, \psi \in \ell^1(\mathbb{Z}^N)$, *we have*

$$\sigma_{\text{ess}}\big(T^<_\varphi + V^<_\psi\big) = \sigma\big(T^<_\varphi + V^<_\psi\big) = \bigcup_{\tau\in\mathbb{T}} \sigma\big(T^{N-1}_{\mu(\tau)\varphi} + V^{N-1}_{\mu(0)\psi}\big).$$

4 Essential spectrum of the fiber Hamiltonians

- For each $\tau \in \mathbb{T}$, $j \in \{2, \ldots, N\}$, $\nu_j(\tau) : \ell^1(\mathbb{Z}^{N-1}) \to \ell^1(\mathbb{Z}^{N-2})$ is defined by

$$[\nu_j(\tau)\rho](z_2, \ldots, z_{j-1}, z_{j+1}, \ldots, z_N) \\ := (\mathscr{F}_j \rho)(z_2, \ldots, z_{j-1}, \tau, z_{j+1}, \ldots, z_N)$$

- $\Sigma_j(\tau, \tau')$, spectrum of the Toeplitz operator (associated to the pair) $\{\mathbb{Z}^{N-2}, (\mathbb{N}^*)^{N-2}\}$)

$$T^{N-2}_{\nu_j(\tau')\mu(\tau)\varphi} + V^{N-2}_{\nu_j(0)\mu(0)\psi}$$

acting in $\ell^2[(\mathbb{N}^*)^{N-2}]$.

Theorem 4.1. *Let* $\varphi, \psi \in \ell^1(\mathbb{Z}^N)$ *be real functions and* $\tau \in \mathbb{T}$. *Then we have*

$$\sigma_{\mathrm{ess}}\big(T^{N-1}_{\mu(\tau)\varphi} + V^{N-1}_{\mu(0)\psi}\big) = \bigcup_{j=2}^{N} \bigcup_{\tau' \in \mathbb{T}} \Sigma_j(\tau, \tau').$$

Outline of the proof. Do the quotient of $\mathcal{T}^{\otimes(N-1)}$ by $\mathscr{K}[\ell^2(\mathbb{N})]^{\otimes(N-1)}$. □

5 Non propagation estimates

We apply the following result which can be formulated rigourously in terms of ideals of the C*-algebra $\mathcal{T}^{<}(\mathbb{Z}^{\mathsf{N}})$ [Amrein, Purice, Măntoiu, 2002].

If κ is a real continuous function with appropriate support, there exists a natural family of multiplication operators $\{\chi_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}_{\mathsf{N}}$ satisfying the following property :

At energies in $\mathrm{supp}(\kappa)$, the system described by H_{N} stays "out of $\mathrm{supp}(\chi_n)$" uniformly in time.

– $\mathrm{supp}(f; H)$, spectral support of a vector f (in a Hilbert space $\mathcal{H}$) w.r.t. a selfadjoint operator H in $\mathcal{H}$
– $\Omega_j(n) := \big\{(y_1, \ldots, y_N) \in \mathbb{Z}^N_< : y_j - y_{j-1} \geq n\big\}$

Proposition 5.1. *Let* $\varphi, \psi \in \ell^1(\mathbb{Z}^N)$ *be real functions and* $j \in \{2, \ldots, N\}$*. Then, for each* $\varepsilon > 0$*, there exists* $n_\varepsilon \in \mathbb{N}$ *such that*

$$\left\|\chi_{\Omega_j(n)} e^{-it\left(T^<_\varphi + V^<_\psi\right)} f\right\| \leq \varepsilon \|f\|$$

for each $n \geq n_\varepsilon$, $t \in \mathbb{R}$, *and each* $f \in \ell^2(\mathbb{Z}^N_<)$ *satisfying*

$$\mathrm{supp}\left(f; T^<_\varphi + V^<_\psi\right) \cap \left[\textstyle\bigcup_{\tau, \tau' \in \mathbb{T}} \Sigma_j(\tau, \tau')\right] = \varnothing.$$

Physical interpretation :

If f is a normalized initial state with energy outside $\cup_{\tau,\tau'\in\mathbb{T}}\Sigma_j(\tau,\tau')$, the decomposition of the system into two clusters of spins pointing up, one ''at the left'', composed of $j-1$ elements, and the other one ''at the right'', composed of $N-j+1$ elements, is highly unprobable *uniformly in time* if the distance n between the clusters is large enough.

6 Current prospects, open problems

- Obtaining more accurate spectral and scattering properties. Mourre estimate.
- Generalization to the lattice $\mathbb{Z}^{\mathfrak{m}}$ in order to treat the $\mathfrak{m}$-dimensional Heisenberg model. In this case, Toeplitz algebras **do not constitute** a suitable mathematical framework.
- Studying directly finite difference operators (Laplacian, adjacency matrix, *etc*) on more general graphs (even convolution operators on locally compact groups).