Ruled strips with asymptotically diverging twisting

Rafael Tiedra de Aldecoa

Pontifical Catholic University of Chile

Metz, November 2021

Joint work with David Krejcirik (Prague)

Table of Contents

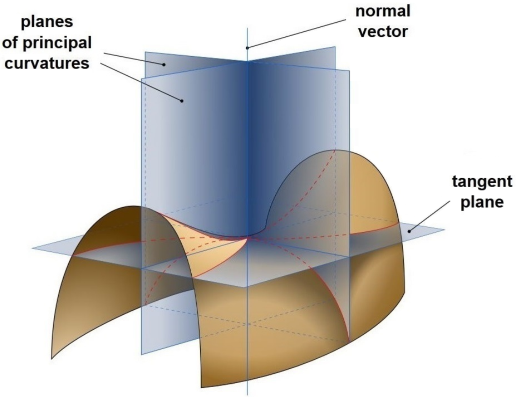
Motivation

Laplacian in tubes (tubular neighbourhoods of submanifolds) have been the subject of intensive study motivated by nanotechnology.

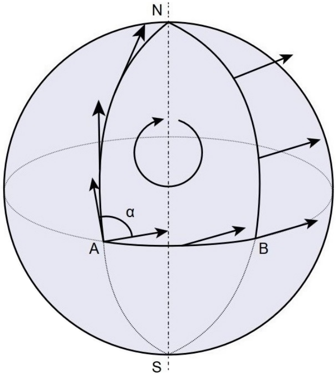
Straight tube

Spectral properties of the Dirichlet Laplacian in unbounded tubes in Euclidean space can be summarised as follows:

- If the principal curvatures of a complete non-compact submanifold $\Sigma \subset \mathbb{R}^{\text{dim}+\text{codim}}$ of dimension dim vanish at infinity and the transport of the tube cross section $\omega \subset \mathbb{R}^{\text{codim}}$ along Σ is asymptotically parallel, then the essential spectrum of the Dirichlet Laplacian is $[E_1, \infty)$, with E_1 the lowest Dirichlet eigenvalue of ω [Krejcirik-Lu 14].



Principal curvatures



Parallel transport

- In the precedent situation, if the transport of ω along Σ is parallel, then the Dirichlet Laplacian in the tube has discrete eigenvalues below E_1 whenever Σ is parabolic (typically dim ≤ 2) and non-trivially curved [Duclos-Exner 95].

Curved tube

Recall that :

11.5. Parabolic manifolds

DEFINITION 11.13. A weighted manifold $(M, \mathbf{g}, \mu)$ is called *parabolic* if any positive superharmonic function on M is constant.

THEOREM 11.14. *Let $(M, \mathbf{g}, \mu)$ be a complete connected weighted manifold. If, for some point $x_0 \in M$,*

$$\int^{\infty} \frac{r dr}{V(x_0, r)} = \infty, \qquad (\bigstar)$$

then M is parabolic.

For example, $(\bigstar)$ holds if $V(x_0, r) \leq C r^2$ for all r large enough.

- Still in the precedent situation, if Σ is totally geodesic and the transport of ω along Σ is not parallel (in which case the tube is *twisted*), then the spectrum of the Dirichlet Laplacian is purely essential [Ekholm-Kovarik-Krejcirik 08].

Twisted square tubes

Recall that :

Definition

A submanifold M of a Riemannian manifold $(\widetilde{M}, \widetilde{g})$ is totally geodesic if any geodesic on the submanifold M with its induced Riemannian metric g is also a geodesic on the Riemannian manifold $(\widetilde{M}, \widetilde{g})$.

Less is known when the transport of the cross section ω along Σ is not asymptotically parallel...

Model

The tube that we consider is

$$\Omega := \mathcal{L}\big(\mathbb{R} \times (a_1, a_2)\big) \subset \mathbb{R}^3 \quad (a_1 < a_2),$$

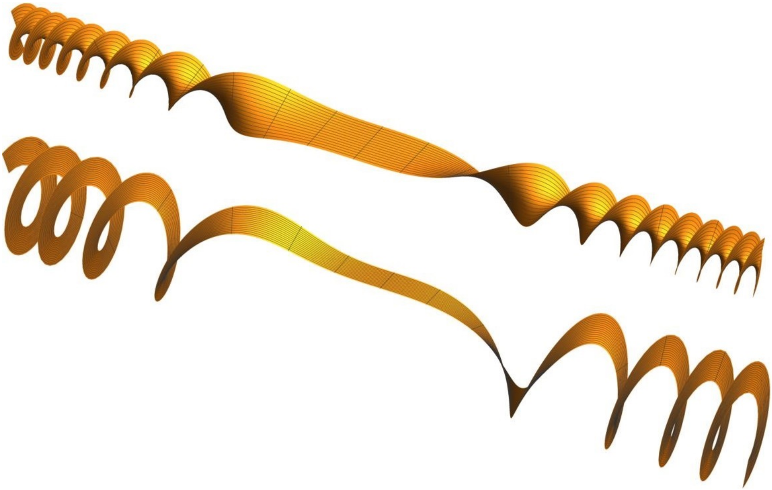
where

$$\mathcal{L} : \mathbb{R}^2 \to \mathbb{R}^3, \quad (s, t) \mapsto \big(s, t\cos\theta(s), t\sin\theta(s)\big)$$

with $\theta : \mathbb{R} \to \mathbb{R}$ a locally Lipschitz function satisfying

$$\lim_{|s|\to\infty} \big|\theta'(s)\big| = \infty.$$

Ω is a ruled strip given by segments (a_1, a_2) translated along a straight line in $\mathbb{R}^3$ with rotation angle θ.



Case $a_1 a_2 \leq 0$ above, case $a_1 a_2 > 0$ below

This corresponds to:

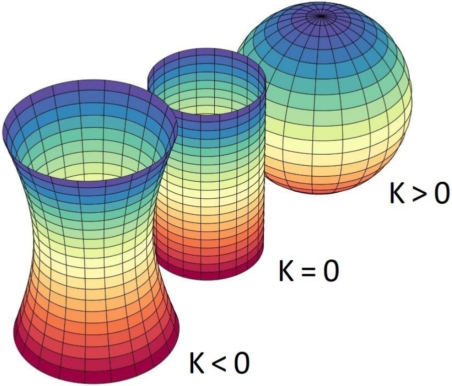
- dim = codim = 1,
- $\Sigma = \mathbb{R}$ parabolic and totally geodesic,
- $\omega = (a_1, a_2)$ symmetric if $a_1 a_2 \leq 0$ or non-symmetric if $a_1 a_2 > 0$,
- transport of ω along Σ far from being parallel (which occurs when $\theta' = 0$) because $\lim_{|s| \to \infty} |\theta'(s)| = \infty$.

The Gauss curvature of Ω

$$K(s,t) = -\frac{\theta'(s)^2}{\left(1+\theta'(s)^2\,t^2\right)^2},$$

tends to zero as $|s| \to \infty$ for every $t \neq 0$.

The Gauss curvature of a surface is the product of its principal curvatures:



The Dirichlet Laplacian $-\Delta_{\mathrm{D}}^{\Omega}$ in Ω is defined as follows:

(i) Ω is identified with the Riemannian submanifold $\big(\mathbb{R} \times (a_1, a_2), G\big)$ of $\mathbb{R}^3$, with G the metric induced by $\mathcal{L}$

$$G := \nabla\mathcal{L} \cdot (\nabla\mathcal{L})^\top = \begin{pmatrix} f^2 & 0 \\ 0 & 1 \end{pmatrix}, \quad f(s,t) := \sqrt{1 + \theta'(s)^2\, t^2}.$$

G^{ij} are the coefficients of the inverse matrix G^{-1}.

(ii) $H := -\Delta_D^\Omega$ is the self-adjoint operator in the Hilbert space

$$\mathcal{H} := \mathsf{L}^2\left(\mathbb{R} \times (a_1, a_2), f(s,t)\,\mathrm{d}s\,\mathrm{d}t\right)$$

given by the closure of the form

$$\begin{aligned}\dot{h}(\Psi) &:= \sum_{i,j=1}^{2} \left(\partial_i \Psi, G^{ij} \partial_j \Psi\right)_{\mathcal{H}} \\ &= \int_{\mathbb{R}\times(a_1,a_2)} f^{-1} |\partial_s \Psi|^2 \,\mathrm{d}s\,\mathrm{d}t + \int_{\mathbb{R}\times(a_1,a_2)} f\,|\partial_t \Psi|^2 \,\mathrm{d}s\,\mathrm{d}t,\end{aligned}$$

where

$$\Psi \in \mathcal{D}(\dot{h}) := C_c^\infty\big(\mathbb{R} \times (a_1, a_2)\big).$$

(iii) In a distributional sense, H acts as

$$H = -\sum_{i,j=1}^{2} |G|^{-1/2}\,\partial_i\,|G|^{1/2}G^{ij}\partial_j = -f^{-1}\,\partial_s\,f^{-1}\,\partial_s - f^{-1}\partial_t\,f\,\partial_t$$

where

$$|G| := \det(G) = f^2.$$

Results

At infinity, the 2D strip Ω looks like a 3D tube of annular cross section

$$A_{r_1,r_2} := \{x \in \mathbb{R}^2 \mid r_1 < |x| < r_2\},$$

where

$$r_1 := \begin{cases} \min\{|a_1|, |a_2|\} & \text{if } a_1 a_2 > 0 \\ 0 & \text{if } a_1 a_2 \leq 0 \end{cases} \quad \text{and} \quad r_2 := \max\{|a_1|, |a_2|\}.$$

Our first theorem shows that this intuition is correct:

First Theorem

If $\lim_{|s|\to\infty} |\theta'(s)| = \infty$, *then*

$$\sigma_{\text{ess}}(H) = [\lambda_1, \infty),$$

with λ_1 *the lowest eigenvalue of the Dirichlet Laplacian in* A_{r_1,r_2}.

Our second theorem shows that there is spectrum below λ_1 if ω is twisted with respect to a point inside ω :

Second Theorem

If $a_1 a_2 \leq 0$, *then* $\inf \sigma(H) < \lambda_1$.

When $\lim_{|s|\to\infty} |\theta'(s)| = \infty$, then λ_1 is the infimum of $\sigma_{\text{ess}}(H)$ by the first theorem. In this case, the second theorem implies the existence of discrete eigenvalues below the essential spectrum :

Corollary

If $\lim_{|s|\to\infty} |\theta'(s)| = \infty$ *and* $a_1 a_2 \leq 0$*, then* $\sigma_{\text{disc}}(H) \cap (0, \lambda_1) \neq \varnothing$*.*

Open question: Can we determine the nature of the essential spectrum of H in the first theorem ? Maybe with Mourre theory ?

Idea of the proof of the first theorem.

(i) We show that $\inf \sigma_{\text{ess}}(H) \geq \lambda_1$ using the fact that

$$\inf \sigma_{\text{ess}}(H) \geq \inf \sigma(H_{\text{ext}}^{\text{N}}),$$

where $H_{\text{ext}}^{\text{N}}$ is the restriction of H to the subspace

$$\mathrm{L}^2\left(\mathbb{R} \setminus [-s_0, s_0] \times (a_1, a_2), f(s,t)\,\mathrm{d}s\,\mathrm{d}t\right), \quad s_0 > 0,$$

with Neumann boundary condition at the segments $\{s = \pm s_0\}$.

(ii) We show the inclusion $\sigma_{\text{ess}}(H) \supset [\lambda_1, \infty)$ using a Weyl criterion for quadratic forms. □

Recall that :

Lemma (Weyl criterion). *A point λ is in the essential spectrum of a self-adjoint operator A if and only if there is a sequence $\psi_n \in \mathfrak{D}(A)$ such that $\|\psi_n\| = 1$, ψ_n converges weakly to 0, and $\|(A-\lambda)\psi_n\| \to 0$.*

Idea of the proof of the second theorem.

We construct a test function $\Psi \in \mathcal{D}(H)$ of the form $\Psi(s,t) = \varphi(s)\psi(t)$ such that

$$\langle \Psi, H\Psi \rangle_{\mathcal{H}} < \lambda_1 \|\Psi\|_{\mathcal{H}}^2.$$

Since

$$\inf \sigma(H) = \inf_{\Psi \in \mathcal{D}(H),\, \|\Psi\|_{\mathcal{H}}=1} \langle \Psi, H\Psi \rangle_{\mathcal{H}},$$

the claim follows. □

Thank you

References

- P. Duclos and P. Exner. Curvature-induced bound states in quantum waveguides in two and three dimensions. *Rev. Math. Phys.*, 1995
- T. Ekholm, H. Kovarik and D. Krejcirik. A Hardy inequality in twisted waveguides. *Arch. Ration. Mech. Anal.*, 2008
- D. Krejcirik. and Z. Lu. Location of the essential spectrum in curved quantum layers. *J. Math. Phys.*, 2014
- D. Krejcirik and R. Tiedra de Aldecoa. Ruled Strips with Asymptotically Diverging Twisting. *Ann. Henri Poincaré*, 2018