Spectral and scattering properties of quantum walks on homogenous trees of odd degree

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Mourre theory in one Hilbert space

- $\mathcal{H}$, Hilbert space with norm $\|\cdot\|$ and scalar product $\langle\cdot,\cdot\rangle$
- $\mathscr{B}(\mathcal{H})$, set of bounded linear operators on $\mathcal{H}$
- $\mathscr{K}(\mathcal{H})$, set of compact operators on $\mathcal{H}$
- U, unitary operator in $\mathcal{H}$ with spectral measure $E^U(\cdot)$, spectrum

$$\sigma(U) \subset \mathbb{S}^1 := \{\, \mathrm{e}^{it} \mid t \in [0, 2\pi) \},$$

  and subspace of absolute continuity $\mathcal{H}_{\mathrm{ac}}(U)$
- A, self-adjoint operator in $\mathcal{H}$ with domain $\mathcal{D}(A)$

Definition

$U \in C^k(A)$ if the map

$$\mathbb{R} \ni t \mapsto \mathrm{e}^{-itA}\, U\, \mathrm{e}^{itA} \in \mathscr{B}(\mathcal{H})$$

is strongly of class C^k.

Intuitively, if $U \in C^k(A)$, then the k-th commutator

$$\Big[\cdots\big[[U, \underbrace{A], A\big], \ldots, A}_{k \text{ times}}\Big]$$

is a well-defined bounded operator.

Definition

$U \in C^{1+\varepsilon}(A)$ for some $\varepsilon \in (0,1)$ if $U \in C^1(A)$ and

$$\big\| \mathrm{e}^{-itA}[U,A]\,\mathrm{e}^{itA} - [U,A]\big\|_{\mathscr{B}(\mathcal{H})} \le \text{Const.}\; t^{\varepsilon} \quad \text{for all } t \in (0,1).$$

One has the inclusions:

$$C^2(A) \subset C^{1+\varepsilon}(A) \subset C^1(A) \subset C^0(A) = \mathscr{B}(\mathcal{H}).$$

Theorem (Fernández-Richard-T. 2013)

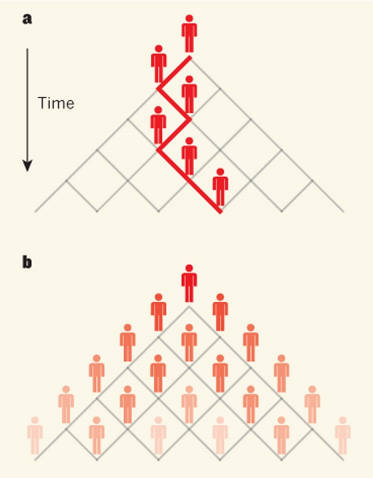
Let $U \in C^{1+\varepsilon}(A)$ and suppose there exist an open set $\Theta \subset \mathbb{S}^1$, $a > 0$ and $K \in \mathscr{K}(\mathcal{H})$ such that

$$E^U(\Theta)\, U^{-1}[A, U]\, E^U(\Theta) \geq a\, E^U(\Theta) + K. \qquad (\bigstar)$$

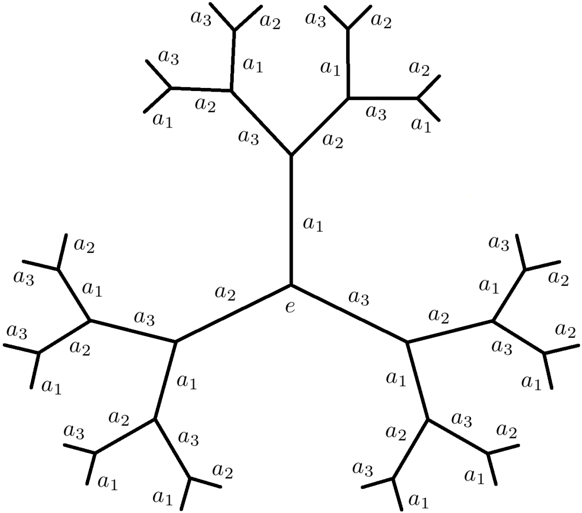
Then U has at most finitely many eigenvalues in Θ, each one of finite multiplicity, and no singular continuous spectrum in Θ.

- The inequality $(\bigstar)$ is called a Mourre estimate for U on Θ.
- The operator A is called a conjugate operator for U on Θ.
- If $K = 0$, then U has purely a.c. spectrum in $\Theta \cap \sigma(U)$.

Quantum walks on homogeneous trees of odd degree $d \geq 3$



Classical random walk (**a**) and quantum walk (**b**) (from nature.com)



Homogeneous tree $\mathcal{T}$ of degree $d = 3$

Even/odd elements of $\mathcal{T}$ defined with the word length $|\cdot|$:

$$\mathcal{T}_{\mathsf{e}} := \left\{ x \in \mathcal{T} \mid |x| \in 2\mathbb{N} \right\} \quad \text{and} \quad \mathcal{T}_{\mathsf{o}} := \left\{ x \in \mathcal{T} \mid |x| \in 2\mathbb{N}+1 \right\}$$

with characteristic functions $\chi_{\mathcal{T}_{\mathsf{e}}}$ and $\chi_{\mathcal{T}_{\mathsf{o}}}$.

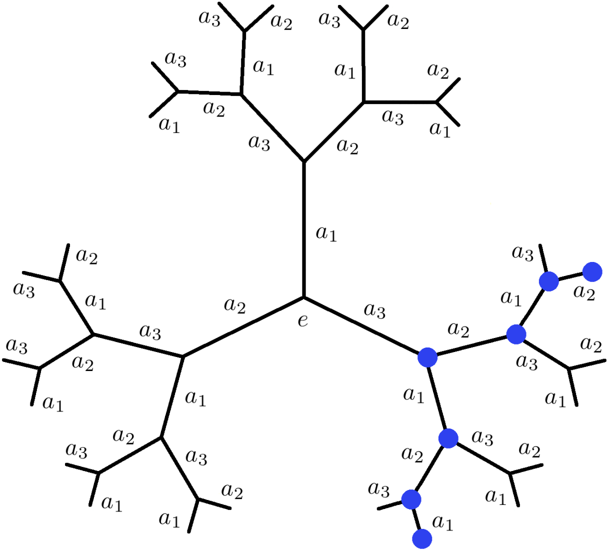
In $\mathcal{H} := \ell^2(\mathcal{T}, \mathbb{C}^d)$ the evolution operator is $U := SC$ with[1]

$$S := \begin{pmatrix} S_{1+1,1+2} & & 0 \\ & \ddots & \\ 0 & & S_{d+1,d+2} \end{pmatrix}, \quad S_{d,d+1} := S_{d,1},\ S_{d+1,d+2} := S_{1,2},$$

$$S_{ij} f := \chi_{\mathcal{T}_{\mathsf{e}}} f(\cdot\, a_i) + \chi_{\mathcal{T}_{\mathsf{o}}} f(\cdot\, a_j), \quad f \in \ell^2(\mathcal{T}) \quad \text{(shift)},$$

$$(C\varphi)(x) := C(x)\varphi(x), \quad \varphi \in \mathcal{H},\ x \in \mathcal{T},\ C(x) \in \mathrm{U}(d) \quad \text{(coin)}.$$

[1] Definitions of [Hamza-Joye 2014] and [Joye-Marin 2014].



Support of the iterates $S_{12}^n \delta_{a_3}$ ($d = 3, n \in \mathbb{Z}$)

C is anisotropic; it converges to an asymptotic coin on each branch of $\mathcal{T}$:

Assumption (Short-range)

For $i = 1, \ldots, d$, there is a diagonal matrix $C_i \in \mathrm{U}(d)$ and $\varepsilon_i > 0$ such that

$$\left\| C(x) - C_i \right\|_{\mathscr{B}(\mathbb{C}^d)} \leq \mathrm{Const.} \left(1 + |x|^2\right)^{-(1+\varepsilon_i)/2} \quad \text{if } x \in \mathcal{T}_i$$

where $\mathcal{T}_i := \left\{ x \in \mathcal{T} \mid \text{the first letter of } x \in \mathcal{T} \text{ is } a_i \right\}$.

Free evolution operator

$\rightsquigarrow$ $U_i := SC_i$ describes the asymptotic behaviour of U on $\mathcal{T}_i$

The assumption also suggests to define the free evolution operator as

$$U_0 := \bigoplus_{k=1}^{d} U_k \quad \text{in} \quad \mathcal{H}_0 := \bigoplus_{k=1}^{d} \mathcal{H},$$

and the identification operator $J : \mathcal{H}_0 \to \mathcal{H}$ as

$$J\Phi := \sum_{k=1}^{d} \chi_k \varphi_k, \quad \Phi = (\varphi_1, \ldots, \varphi_d) \in \mathcal{H}_0,$$

with

$$\chi_1 := \chi_{\mathcal{T}_1 \cup \{e\}}, \quad \chi_k := \chi_{\mathcal{T}_k} \ (k \geq 2).$$

For $i \neq j$, define the modified word length

$$|\cdot|_{i,j} : \mathcal{T} \to \mathbb{N}, \quad x \mapsto |x|_{i,j} := \left|x_{i,j}^{-1} x\right|,$$

with $x_{i,j}$ the longest word in the reduced representation of x, starting from the left and ending with a letter different from a_i or a_j.

Example

If $x = a_1 a_2 a_3 a_2$, then $x_{1,2} = a_1 a_2 a_3$, and if $x = a_1 a_2 a_1 a_2$, then $x_{1,2} = e$.

Lemma (Conjugate operator for S_{ij})

Take $i \neq j$. Then the operator

$$A_{i,j} f := S_{i,j}^{-1} \big[|\cdot|_{i,j}^2, S_{i,j} \big] f, \quad f \in C_c(\mathcal{T}),$$

is essentially self-adjoint in $\ell^2(\mathcal{T})$, with closure also denoted by A_{ij}, and $S_{ij} \in C^\infty(A_{ij})$ with

$$S_{ij}^{-1}[A_{ij}, S_{ij}] = 2 \quad \textit{(Mourre estimate for } S_{ij} \textit{ on } \Theta = \mathbb{S}^1\textit{)}.$$

Idea of the proof.

Direct calculations on the dense set $C_c(\mathcal{T})$.[a] □

[a] Reminiscent of the relations

$$\mathtt{A} := \mathtt{S}^{-1}[\mathtt{X}^2, \mathtt{S}] \quad \text{and} \quad \mathtt{S}^{-1}[\mathtt{A}, \mathtt{S}] = 2,$$

with $\mathtt{S}$ the bilateral shift on $\ell^2(\mathbb{Z})$ and $\mathtt{X}$ the position operator on $\ell^2(\mathbb{Z})$.

[Fernández-Richard-T. 2013] implies that S_{ij} has purely a.c. spectrum. But more can be said:

The relation $S_{ij}^{-1}[A_{ij}, S_{ij}] = 2$ implies by functional calculus the imprimitivity relation

$$\mathrm{e}^{isA_{ij}}\,\gamma(S_{ij})\,\mathrm{e}^{-isA_{ij}} = \gamma(\mathrm{e}^{2is}\,S_{ij}), \quad s \in \mathbb{R},\ \gamma \in C(\mathbb{S}^1).$$

This and Mackey's imprimitivity theorem implies that S_{ij} is unitarily equivalent to a multiplication operator with purely a.c. spectrum which covers $\mathbb{S}^1$.

(think of a version of Stone-von Neumann theorem on $\mathbb{S}^1$...)

What precedes + direct sums + diagonality of the C_i's implies:

Proposition (Spectral properties of U_0)

Let

$$A_0 := \bigoplus_{k=1}^{d} \widetilde{A} \quad \text{with} \quad \widetilde{A} := \begin{pmatrix} A_{1+1,1+2} & & & 0 \\ & A_{2+1,2+2} & & \\ & & \ddots & \\ 0 & & & A_{d+1,d+2} \end{pmatrix}.$$

(a) *A_0 is essentially self-adjoint in $\mathcal{H}_0$, with closure also denoted by A_0.*

(b) *$U_0 \in C^\infty(A_0)$ with $U_0^{-1}[A_0, U_0] = 2$, and U_0 satisfies the imprimitivity relation*

$$\mathrm{e}^{isA_0}\,\gamma(U_0)\,\mathrm{e}^{-isA_0} = \gamma(\mathrm{e}^{2is}\,U_0), \quad s \in \mathbb{R},\ \gamma \in C(\mathbb{S}^1).$$

(c) *U_0 is unitarily equivalent to a multiplication operator with purely a.c. spectrum covering the whole unit circle $\mathbb{S}^1$.*

Mourre theory in two Hilbert spaces

Using [Richard-Suzuki-T. 2018], we get the general perturbation result:

Theorem

Let U_0, U be unitary operators in $\mathcal{H}_0, \mathcal{H}$, let A_0 be self-adjoint in $\mathcal{H}_0$, let $J \in \mathscr{B}(\mathcal{H}_0, \mathcal{H})$, and assume

(i) *there is a $\mathscr{D} \subset \mathcal{D}(A_0 J^*) \subset \mathcal{H}$ such that $J A_0 J^* \upharpoonright \mathscr{D}$ is essentially self-adjoint, with closure denoted by A,*

(ii) $U_0 \in C^1(A_0)$,

(iii) *compacity conditions relating U_0, U, A_0, J.*

Then $U \in C^1(A)$, and A is a conjugate operator for U on $\Theta \subset \mathbb{S}^1$ if A_0 is a conjugate operator for U_0 on Θ.

Full evolution operator

Using the short-range assumption, we get that $V := JU_0 - UJ$ is trace class, that $U \in C^{1+\varepsilon}(A)$, and the hypotheses of the last theorem. Thus:

Theorem (Spectral properties of U)

U has at most finitely many eigenvalues, each one of finite multiplicity, and no singular continuous spectrum.

(first spectral result for quantum walks on trees with position-dependent coin)

Wave operators

Theorem (Completeness, version 1)

The wave operators $W_\pm(U, U_0, J) : \mathcal{H}_0 \to \mathcal{H}$ *given by*

$$W_\pm(U, U_0, J) := \operatorname*{s-lim}_{n \to \pm\infty} U^{-n} J U_0^n$$

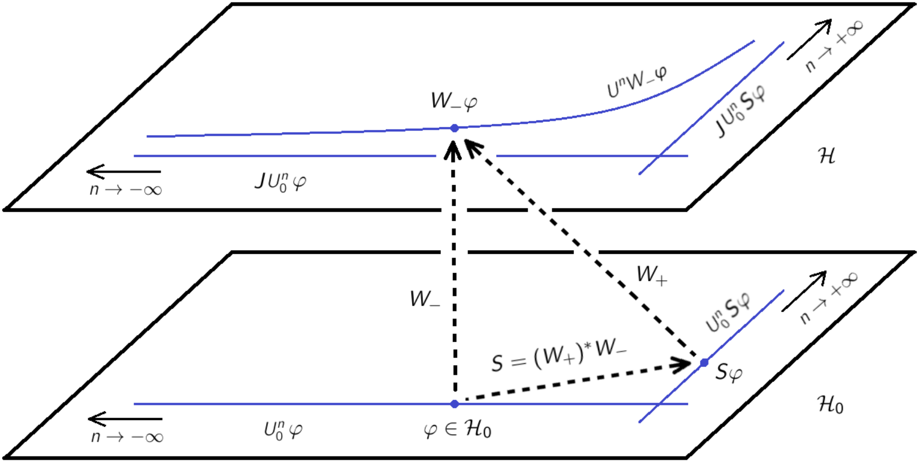
exist and are complete, that is, $\mathrm{Ran}\left(W_\pm(U, U_0, J)\right) = \mathcal{H}_{\mathrm{ac}}(U)$.

Idea of the proof.

$W_\pm(U, U_0, J)$ exist because V is trace class. For the completeness, the idea is to note that $\mathcal{H}_{\mathrm{ac}}(U) \supset \mathrm{Ran}\left(W_\pm(U, U_0, J)\right) \supset \mathrm{Ran}(a_\pm)$ with

$$a_\pm \varphi := W_\pm(U, U_0, J)(\varphi, \ldots, \varphi), \quad \varphi \in \mathcal{H},$$

and show that $\mathrm{Ran}(a_\pm) = \mathcal{H}_{\mathrm{ac}}(U)$. □


$U^n W_- \varphi$
$W_- \varphi$
$J U_0^n S \varphi$
$n \to +\infty$
$\mathcal{H}$
$n \to -\infty$
$J U_0^n \varphi$
W_+
W_-
$U_0^n S \varphi$
$n \to +\infty$
$S = (W_+)^* W_-$
$S \varphi$
$\mathcal{H}_0$
$n \to -\infty$
$U_0^n \varphi$
$\varphi \in \mathcal{H}_0$

A direct calculation shows that $a_\pm = \operatorname*{s-lim}_{n\to\pm\infty} U^{-n}\big(\sum_{k=1}^d \chi_k C_k\big)^n S^n$. Thus:

Corollary (Completeness, version 2)

The operators $a_\pm : \mathcal{H} \to \mathcal{H}$ *given by*

$$a_\pm := \operatorname*{s-lim}_{n\to\pm\infty} U^{-n} J_n S^n, \quad J_n := \big(\sum_{k=1}^d \chi_k C_k\big)^n,$$

exist, are isometric, and satisfy $\mathrm{Ran}(a_\pm) = \mathcal{H}_{\mathrm{ac}}(U)$.

In other words: If one uses $(\sum_{k=1}^d \chi_k C_k)^n$ as time-dependent identification operators, then the "trivial" shift S can be used as a free evolution operator.

Corollary

One has $\sigma_{\mathrm{ac}}(U) = \mathbb{S}^1$.

Idea of the proof.

Follows from the fact that $a_\pm$ is unitary from $\mathcal{H} = \mathcal{H}_{\mathrm{ac}}(S)$ to $\mathcal{H}_{\mathrm{ac}}(U)$. □

Thus, the spectrum of U covers $\mathbb{S}^1$ and is purely a.c., outside a finite set where U may have eigenvalues of finite multiplicity.

Finally:

Theorem (Completeness, version 3)

The wave operators $W_\pm(U, \widetilde{U}_0) : \mathcal{H} \to \mathcal{H}$ *given by*

$$W_\pm(U, \widetilde{U}_0) := \operatorname*{s-lim}_{n \to \pm\infty} U^{-n} (\widetilde{U}_0)^n, \quad \widetilde{U}_0 := S \sum_{k=1}^{d} \chi_k C_k,$$

exist, are isometric, and are complete, that is, $\mathrm{Ran}(W_\pm(U, \widetilde{U}_0)) = \mathcal{H}_{\mathrm{ac}}(U)$.

Idea of the proof.

Follows from the facts that $\sum_{k=1}^{d} \chi_k C_k$ is diagonal (but not constant) and $\widetilde{U}_0 - U$ trace class. □

Open problems

1. What are the initial subspaces of $W_{\pm}(U, U_0, J)$?
2. What are the asymptotic velocity operators of $U_i = SC_i$?
3. The case of coin operators that converge at infinity along other partitions of the tree into subtrees ?
4. The case of coin operators that converge at infinity to arbitrary constant unitary matrices ? Helgason-Fourier transform ?
5. The case of trees of even degree ?
6. The case of rooted trees ?

Thank you

References

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