Commutator methods for unitary operators

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Contents

1 Motivation

Develop commutator methods for unitary operators up to an optimality equivalent to the one for self-adjoint operators.

2 Commutators methods for self-adjoint operators

- $\mathcal{H}$, Hilbert space with norm $\|\cdot\|$ and scalar product $\langle\cdot,\cdot\rangle$
- $\mathscr{B}(\mathcal{H})$, set of bounded linear operators on $\mathcal{H}$
- $\mathscr{K}(\mathcal{H})$, set of compact operators on $\mathcal{H}$
- A, H, self-adjoint operators in $\mathcal{H}$ with domains $\mathcal{D}(A), \mathcal{D}(H)$, spectral measures $E^A(\cdot), E^H(\cdot)$ and spectra $\sigma(A), \sigma(H)$

Definition 1.1. *An operator $S \in \mathscr{B}(\mathcal{H})$ satisfies $S \in C^k(A)$ if*

$$\mathbb{R} \ni t \mapsto \mathrm{e}^{-itA} S \, \mathrm{e}^{itA} \in \mathscr{B}(\mathcal{H})$$

is strongly of class C^k.

$S \in C^1(A)$ if and only if

$$\big|\big\langle A\varphi, S\varphi\big\rangle - \big\langle \varphi, SA\varphi\big\rangle\big| \leq \mathrm{Const.} \, \|\varphi\|^2 \quad \text{for all } \varphi \in \mathcal{D}(A).$$

The operator corresponding to the continuous extension of the quadratic form is denoted by $[S, A]$, and one has

$$[iS, A] = \text{s-} \frac{\mathrm{d}}{\mathrm{d}t} \, \mathrm{e}^{-itA} S \, \mathrm{e}^{itA} \bigg|_{t=0} \in \mathscr{B}(\mathcal{H}).$$

Example 1.2. *Let Q and P be the position and differentiation operators in $\mathsf{L}^2(\mathbb{R})$; that is,*

$$(Q\varphi)(x) := x\,\varphi(x), \quad (P\varphi)(x) := -i\,\varphi'(x), \quad \varphi \in \mathscr{S}(\mathbb{R}) \subset \mathsf{L}^2(\mathbb{R}),$$

and take $f \in \mathsf{L}^\infty(\mathbb{R})$ an a.c. function with $f' \in \mathsf{L}^\infty(\mathbb{R})$.

Then, $f(Q) \in C^1(P)$ with

$$\big[P, f(Q)\big] = -i\,f'(Q).$$

(calculations are done on the core $\mathscr{S}(\mathbb{R})$ . . .)

Definition 1.3. *$S \in C^{1+0}(A)$ if $S \in C^1(A)$ and*

$$\int_0^1 \frac{\mathrm{d}t}{t} \left\| \mathrm{e}^{-itA}[A,S]\,\mathrm{e}^{itA} - [A,S] \right\| < \infty.$$

If we regard $C^1(A)$, $C^{1+0}(A)$ and $C^2(A)$ as subspaces of $\mathscr{B}(\mathcal{H})$, then we have the inclusions

$$C^2(A) \subset C^{1+0}(A) \subset C^1(A) \subset C^0(A) \equiv \mathscr{B}(\mathcal{H}).$$

(in the example, $f(Q) \in C^{1+0}(P)$ if f' is Dini-continuous . . .)

Definition 1.4. *A self-adjoint operator H is of class $C^k(A)$ if $(H-z)^{-1} \in C^k(A)$ for some $z \in \mathbb{C} \setminus \sigma(H)$.*

If H is of class $C^1(A)$, then

$$\big[A, (H-z)^{-1}\big] = (H-z)^{-1}[H, A](H-z)^{-1},$$

with $[H, A] \in \mathscr{B}\big(\mathcal{D}(H), \mathcal{D}(H)^*\big)$ the operator corresponding to the continuous extension to $\mathcal{D}(H)$ of the quadratic form

$$\mathcal{D}(H) \cap \mathcal{D}(A) \ni \varphi \mapsto \big\langle H\varphi, A\varphi\big\rangle - \big\langle A\varphi, H\varphi\big\rangle \in \mathbb{C}.$$

Spectral result of É. Mourre (and W. Amrein, A. Boutet de Monvel, V. Georgescu, J. Sahbani, . . .)

Theorem 1.5. *Let H be of class $C^{1+0}(A)$. Assume there exist an open bounded set $I \subset \mathbb{R}$, a number $a > 0$ and $K \in \mathscr{K}(\mathcal{H})$ such that*

$$E^H(I)\,[iH, A]E^H(I) \geq a\,E^H(I) + K. \qquad (\bigstar)$$

Then, H has at most finitely many eigenvalues in I (multiplicities counted), and H has no singular continuous spectrum in I.

- The inequality (★) is called a Mourre estimate.
- If $K = 0$, then H is purely absolutely continuous in $I \cap \sigma(H)$.
- There is no spectral gap assumption on H.
- The proof relies on differential inequalities for the resolvent $(H - z)^{-1}$ of H which lead to boundary estimates for the map $z \mapsto (H - z)^{-1}$ (other proofs are available in the case $C^2(A)$, see [Golénia/Jecko 2007], [Gérard 2008], $\ldots$).

3 Commutators methods for unitary operators

Let U be a unitary operator in $\mathcal{H}$, then

- similar results were known under the assumption $U \in C^2(A)$,
- the proofs relied once more on differential inequalities for "resolvents" of U [Astaburuaga et al. 2006].

We want the weaker assumption $U \in C^{1+0}(A)$ and a simpler proof.

What we obtain at the end of the day:

Theorem 1.6 ([Fernández/Richard/T. 2013]). *Let* $U \in C^{1+0}(A)$*. Assume there exist an open set* $\Theta \subset \mathbb{T}$*, a number* $a > 0$ *and* $K \in \mathscr{K}(\mathcal{H})$ *such that*

$$E^U(\Theta)\, U^*[A, U]\, E^U(\Theta) \geq a\, E^U(\Theta) + K.$$

Then, U *has at most finitely many eigenvalues in* Θ *(multiplicities counted), and* U *has no singular continuous spectrum in* Θ*.*

Sketch of the proof (a)

Why the "commutator" $U^*[A,U]$ is the right expression to consider ?

Imagine that $U = \mathrm{e}^{-iH}$ with $H \in C^1(A)$, then one would have

$$U^*[A,U] = \int_0^1 \mathrm{d}s\ \mathrm{e}^{isH}\,[iH,A]\,\mathrm{e}^{-isH}\,.$$

So, positivity of $[iH,A]$ would lead to positivity of $U^*[A,U]$ and vice versa.

(the idea of using $U^*[A,U]$ dates back to Putnam in the 60's)

Sketch of the proof (b)

If $U \in C^1(A)$ and

$$E^U(\Theta)\,U^*[A,U]\,E^U(\Theta) \geq a\,E^U(\Theta) + K,$$

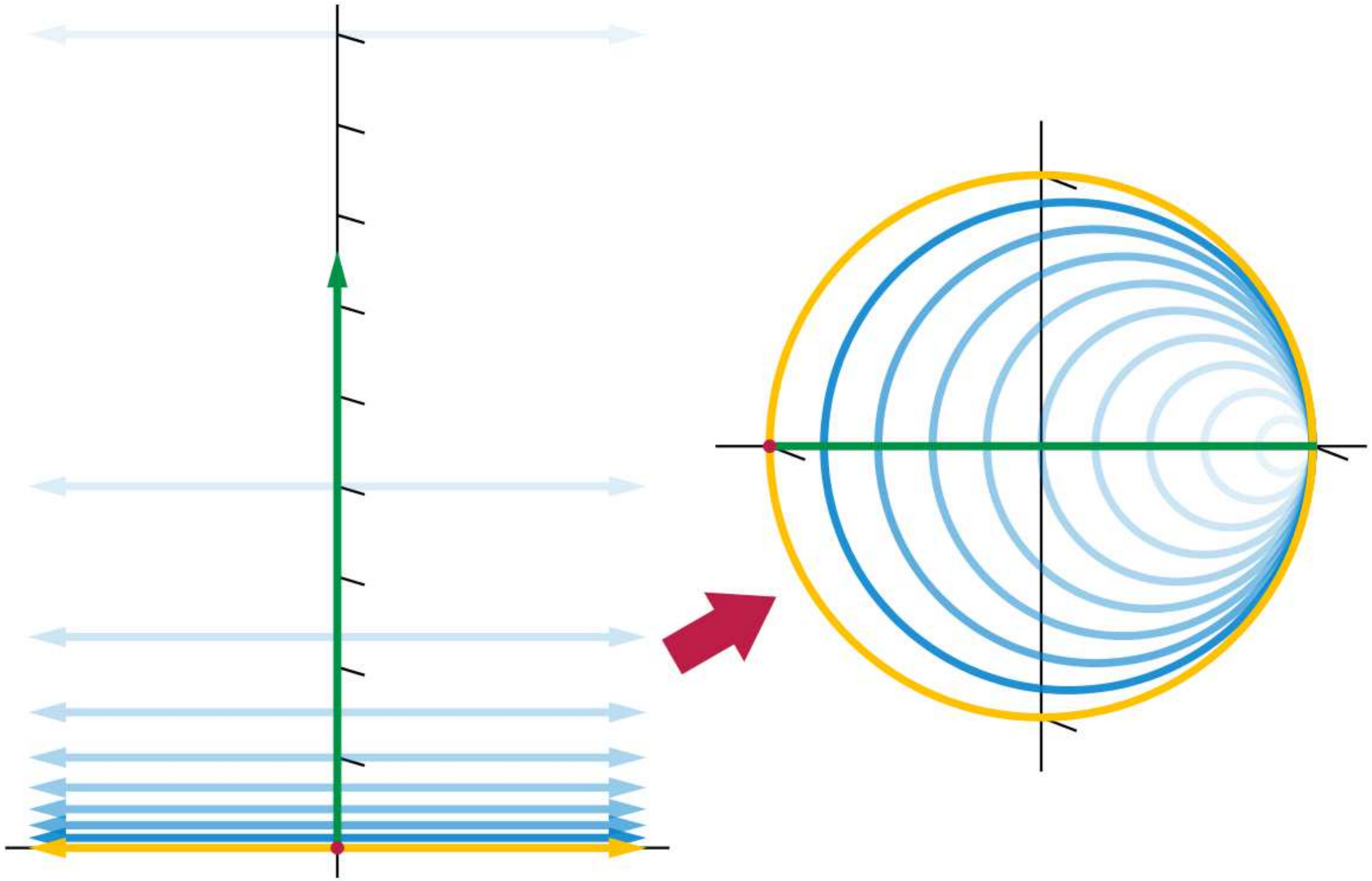
then the Virial theorem for unitary operators implies that U has at most finitely many eigenvalues in Θ (multiplicities counted).

Thus,

- there exists $\theta \in \Theta$ which is not an eigenvalue of U and the range $\mathsf{Ran}(1-\bar{\theta}U)$ of $1-\bar{\theta}U$ is dense in $\mathcal{H}$,
- the Cayley transform of U at the point θ; namely,

$$H_\theta := -i\,(1+\bar{\theta}U)(1-\bar{\theta}U)^{-1}, \quad \mathcal{D}(H_\theta) := \mathsf{Ran}(1-\bar{\theta}U),$$

is a self-adjoint operator.



Cayley transform of $\mathbb{R}$ (for $\theta = -i$)

Sketch of the proof (c)

One has

$$(H_\theta - i)^{-1} = -\frac{1}{2i}(1-\bar{\theta}U) \ \text{ in } \ \mathscr{B}(\mathcal{H})$$

and

$$[iH_\theta, A] = 2\big\{(1-\bar{\theta}U)^{-1}\big\}^* U^*[A,U](1-\bar{\theta}U)^{-1} \ \text{ in } \mathscr{B}\big(\mathcal{D}(H_\theta), \mathcal{D}(H_\theta)^*\big).$$

Thus,

- $U \in C^{1+0}(A)$ and positivity of $U^*[A,U]$ in $\Theta \subset \mathbb{T}$ implies H_θ of class $C^{1+0}(A)$ and positivity of $[iH_\theta, A]$ in $I \subset \mathbb{R}$,
- H_θ of class $C^{1+0}(A)$ and positivity of $[iH_\theta, A]$ in I implies spectral spectral properties of H_θ in I,
- spectral properties of H_θ in I implies spectral properties of U in Θ.

No need to re-do any proof with differential inequalities; we just use the Cayley transform and the pre-existing self-adjoint theory.

We have the following perturbation result:

Corollary 1.7 ([Fernández/Richard/T. 2013])**.** *Let* U, V *be unitary, with* $U, V \in C^{1+0}(A)$*. Assume there exist an open set* $\Theta \subset \mathbb{T}$*, a number* $a > 0$ *and* $K \in \mathscr{K}(\mathcal{H})$ *such that*

$$E^U(\Theta)\,U^*[A,U]\,E^U(\Theta) \geq a\,E^U(\Theta) + K. \qquad (\bigstar\bigstar)$$

Suppose also that $(V-1) \in \mathscr{K}(\mathcal{H})$*. Then,* VU *has at most finitely many eigenvalues in* Θ *(multiplicities counted), and* VU *has no singular continuous spectrum in* Θ*.*

- The Mourre estimate $(\bigstar\bigstar)$ is expressed only in terms of U (V is the perturbation).
- UV and VU are unitarily equivalent since $UV = U(VU)U^*$.

4 Perturbations of the Schrödinger free evolution

The free evolution $U_t := \mathrm{e}^{-itP^2}$ in $\mathsf{L}^2(\mathbb{R}^d)$ satisfies for each $t > 0$

$$\sigma(U_t) = \sigma_{\mathrm{ac}}(U_t) = \mathbb{T}.$$

The operator

$$A := \frac{1}{2}\Big\{\big(P^2+1\big)^{-1} P \cdot Q + Q \cdot P \big(P^2+1\big)^{-1}\Big\}$$

is essentially self-adjoint on $\mathscr{S}(\mathbb{R}^d)$, and $U \in C^1(A)$ with

$$(U_t)^*[A, U_t] = 2tP^2(P^2+1)^{-1}.$$

(further commutations on $\mathscr{S}(\mathbb{R}^d)$ show that $U \in C^2(A) \subset C^{1+0}(A)$)

If closure$(\Theta) \cap \{1\} = \varnothing$, there exists $\delta > 0$ such that

$$E^{U_t}(\Theta)\,(U_t)^*[A, U_t]\,E^{U_t}(\Theta) \geq 2t\,\delta(\delta+1)^{-1}E^{U_t}(\Theta).$$

So, all the assumptions for U_t are satisfied, and we have:

Lemma 1.8. *If $V \in C^{1+0}(A)$ with $(V-1) \in \mathscr{K}(\mathcal{H})$, then the eigenvalues of VU_t outside $\{1\}$ are of finite multiplicity and can accumulate only at $\{1\}$. Furthermore, VU_t has no singular continuous spectrum.*

(This extends previous results on the Schrödinger free evolution U_t perturbed by periodic kicks; for instance for $V = \mathrm{e}^{iB}$ with $B = B^*$ of finite rank.)

5 Cocycles over irrational rotations

Let $\mathcal{H} := \mathsf{L}^2\big([0,1)\big)$ with addition modulo 1 on $[0,1)$, take $\theta \in \mathbb{R} \setminus \mathbb{Q}$, and let $f : [0,1) \to \mathbb{R}$ be measurable.

Define for $\varphi \in \mathcal{H}$ and almost every $x \in [0,1)$

$$(V\varphi)(x) := \mathrm{e}^{2\pi i x}\,\varphi(x) \quad \text{and} \quad (U\varphi)(x) := \mathrm{e}^{2\pi i f(x)}\,\varphi([x+\theta]),$$

with $[y] := y$ modulo 1.

(V is the unitary multiplication operator by the variable in $\mathbb{T}$ and U is the unitary operator associated with the "cocycle" f over the rotation by θ.)

U and V satisfy the commutation relation of the irrational rotation algebra:

$$UV = \mathrm{e}^{2\pi i\theta} VU.$$

Thus, U and V have full spectrum

$$\sigma(U) = \sigma(V) = \mathbb{T},$$

and $\sigma(U)$ has uniform multiplicity and is either purely punctual, purely singular continuous or purely Lebesgue [Helson 1986].

We treat the case where

$$f := m\,\mathrm{id} + h,$$

with $m \in \mathbb{N}^*$, $\mathrm{id} : [0,1) \to [0,1)$ the identity function and $h : [0,1) \to \mathbb{R}$ an a.c. function satisfying $h(0) = h(1)$.

Let P be the differentiation operator on $[0,1)$; namely,

$$P\varphi := -i\,\varphi', \quad \varphi \in \mathcal{D}(P) := \big\{\varphi \in \mathcal{H} \mid \varphi \text{ a.c., } \varphi' \in \mathcal{H} \text{ and } \varphi(0) = \varphi(1)\big\}.$$

If $h' \in \mathsf{L}^\infty\big([0,1)\big)$, one has for $\varphi \in \mathcal{D}(P)$

$$\big\langle P\varphi, U\varphi\big\rangle - \big\langle\varphi, UP\varphi\big\rangle = \big\langle\varphi, 2\pi\big(m + h'\big)U\varphi\big\rangle,$$

and thus $U \in C^1(P)$ with $U^*[P,U] = 2\pi\big(m + h'([\,\cdot - \theta])\big)$.

Thus, by imposing some condition on the size of h', one would obtain that $U^*[P,U]$ is strictly positive (as did [Kushnirenko 1974] in another set-up).

But, one can obtain such a positivity without any condition on the size of h' . . .

Consider the self-adjoint operator

$$P_n := \frac{1}{n}\sum_{j=0}^{n-1} U^{-j} P U^{j}.$$

Then, $U \in C^1(P_n)$ with

$$\begin{aligned} U^*\big[P_n, U\big] &= U^* \frac{1}{n}\sum_{j=0}^{n-1} U^{-j}[P,U]\,U^{j} \\ &= U^* \frac{2\pi}{n}\sum_{j=0}^{n-1} U^{-j}\big(m + h'\big)U^{j+1} \\ &= 2\pi\left(m + \frac{1}{n}\sum_{j=1}^{n} h'([\,\cdot - j\theta])\right). \end{aligned}$$

But, the unique ergodicity of the rotation by θ implies that

$$\lim_{n\to\infty}\left\|\frac{1}{n}\sum_{j=1}^{n}h'([\,\cdot\,-j\theta])\right\|_{\mathrm{L}^\infty([0,1))}=\int_0^1 \mathrm{d}x\,h'(x)=h(1)-h(0)=0.$$

Thus, $\left|\frac{1}{n}\sum_{j=1}^{n}h'([\,\cdot\,-j\theta])\right|$ is as small as desired and

$$U^*[P_n,U]\equiv 2\pi\left(m+\frac{1}{n}\sum_{j=1}^{n}h'([\,\cdot\,-j\theta])\right)$$

is strictly positive if n large enough. Moreover, if h' is Dini-continuous, then $U\in C^{1+0}(P_n)$ and U satisfies all the required assumptions.

So, U has a purely Lebesgue spectrum equal to $\mathbb{T}$.

6 References

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