Spectral analysis of quantum walks with an anisotropic coin

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Commutators in one Hilbert space

- $\mathcal{H}$, Hilbert space with norm $\|\cdot\|$ and scalar product $\langle\cdot,\cdot\rangle$
- $\mathscr{B}(\mathcal{H})$, set of bounded linear operators on $\mathcal{H}$
- $\mathscr{K}(\mathcal{H})$, set of compact operators on $\mathcal{H}$
- U, unitary operator in $\mathcal{H}$ with spectral measure $E^U(\cdot)$ and spectrum

$$\sigma(U) \subset \mathbb{S}^1 := \{\, \mathrm{e}^{i\gamma} \mid \gamma \in [0, 2\pi) \}$$

- A, self-adjoint operator in $\mathcal{H}$ with domain $\mathcal{D}(A)$

Definition

$U \in C^k(A)$ if the map

$$\mathbb{R} \ni t \mapsto \mathrm{e}^{-itA}\, U\, \mathrm{e}^{itA} \in \mathscr{B}(\mathcal{H})$$

is strongly of class C^k.

$U \in C^1(A)$ if and only if

$$\big|\langle \varphi, U A\varphi\rangle - \langle A\varphi, U\varphi\rangle\big| \leq \text{Const.}\, \|\varphi\|^2 \quad \text{for all } \varphi \in \mathcal{D}(A).$$

The bounded operator associated to the continuous extension of the quadratic form is written $[U, A]$, and

$$[iU, A] = \text{s-}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{e}^{-itA}\, U\, \mathrm{e}^{itA} \in \mathscr{B}(\mathcal{H}).$$

Definition

$U \in C^{1+\varepsilon}(A)$ for some $\varepsilon \in (0,1)$ if $U \in C^1(A)$ and

$$\left\| \mathrm{e}^{-itA}[U,A]\,\mathrm{e}^{itA} - [U,A] \right\|_{\mathscr{B}(\mathcal{H})} \leq \text{Const.}\ t^{\varepsilon} \quad \text{for all } t \in (0,1).$$

One has the inclusions :

$$C^2(A) \subset C^{1+\varepsilon}(A) \subset C^1(A) \subset \mathscr{B}(\mathcal{H}).$$

Theorem (Fernández-Richard-T. 2013)

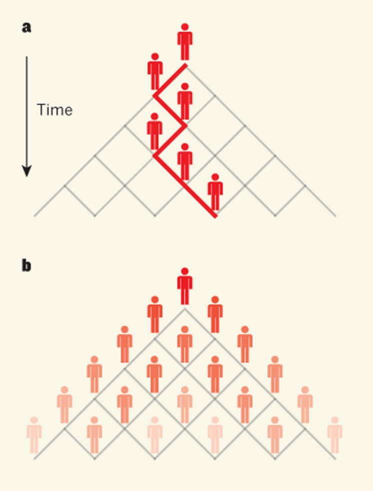
Let $U \in C^{1+\varepsilon}(A)$ and suppose there exist an open set $\Theta \subset \mathbb{S}^1$, $a > 0$ and $K \in \mathscr{K}(\mathcal{H})$ such that

$$E^U(\Theta)\, U^{-1}[A, U]\, E^U(\Theta) \geq a\, E^U(\Theta) + K. \qquad (\bigstar)$$

Then, U has at most finitely many eigenvalues in Θ, each one of finite multiplicity, and U has no singular continuous spectrum in Θ.

- The inequality $(\bigstar)$ is called a Mourre estimate for U on Θ.
- The operator A is called a conjugate operator for U on Θ.
- If $K = 0$, then U has purely absolutely continuous spectrum in $\Theta \cap \sigma(U)$.
- In fact, one obtains a limiting absorption principle (resolvent estimate) under the hypotheses of the theorem.

Quantum walks with an anisotropic coin



Classical random walk (**a**) and quantum walk (**b**) (from nature.com)

In the Hilbert space

$$\mathcal{H} := \ell^2(\mathbb{Z}, \mathbb{C}^2) = \left\{ \Psi : \mathbb{Z} \to \mathbb{C}^2 \mid \sum_{x \in \mathbb{Z}} \|\Psi(x)\|^2_{\mathbb{C}^2} < \infty \right\},$$

the evolution operator of the quantum walk is

$$U := SC$$

with

$$(S\Psi)(x) := \begin{pmatrix} \Psi^{(0)}(x+1) \\ \Psi^{(1)}(x-1) \end{pmatrix}, \quad \Psi = \begin{pmatrix} \Psi^{(0)} \\ \Psi^{(1)} \end{pmatrix} \in \mathcal{H},\ x \in \mathbb{Z}, \qquad \text{(shift)}$$

$$(C\Psi)(x) := C(x)\Psi(x), \quad \Psi \in \mathcal{H},\ x \in \mathbb{Z},\ C(x) \in \mathrm{U}(2). \qquad \text{(coin)}$$

The operator U is unitary because S and C are unitary.

C is short-range and anisotropic at infinity:

Assumption (Anisotropic coin)

There exist $C_\ell, C_{\mathrm{r}} \in \mathrm{U}(2)$, $\kappa_\ell, \kappa_{\mathrm{r}} > 0$, *and* $\varepsilon_\ell, \varepsilon_{\mathrm{r}} > 0$ *such that*

$$\begin{aligned} \left\| C(x) - C_\ell \right\|_{\mathscr{B}(\mathbb{C}^2)} &\le \kappa_\ell \, |x|^{-1-\varepsilon_\ell} \quad \text{if } x < 0 \\ \left\| C(x) - C_{\mathrm{r}} \right\|_{\mathscr{B}(\mathbb{C}^2)} &\le \kappa_{\mathrm{r}} \, |x|^{-1-\varepsilon_{\mathrm{r}}} \quad \text{if } x > 0, \end{aligned}$$

with indexes ℓ *for "left" and* r *for "right".*

Quantum walks satisfying this are called quantum walks with an anisotropic coin.

They include one-defect models, two-phase quantum walks, and topological phase quantum walks as special cases.

Asymptotic evolution operators

The assumption furnishes two unitary operators

$$U_\ell := S\,C_\ell \quad \text{and} \quad U_{\mathrm{r}} := S\,C_{\mathrm{r}}$$

describing the asymptotic behaviour of U on the left and on the right.

We use the parametrisation

$$C_\star = \mathrm{e}^{i\delta_\star/2} \begin{pmatrix} a_\star\,\mathrm{e}^{i(\alpha_\star-\delta_\star/2)} & b_\star\,\mathrm{e}^{i(\beta_\star-\delta_\star/2)} \\ -b_\star\,\mathrm{e}^{-i(\beta_\star-\delta_\star/2)} & a_\star\,\mathrm{e}^{-i(\alpha_\star-\delta_\star/2)} \end{pmatrix}, \quad \star = \ell, \mathrm{r},$$

with $a_\star, b_\star \in [0,1]$ such that $a_\star^2 + b_\star^2 = 1$, and $\alpha_\star, \beta_\star, \delta_\star \in (-\pi, \pi]$.

Lemma (Spectrum of $U_\star$)

(a) *If $a_\star = 0$, then $U_\star$ has pure point spectrum*

$$\sigma(U_\star) = \sigma_{\mathsf{p}}(U_\star) = \left\{ i\,\mathrm{e}^{i\delta_\star/2}, -i\,\mathrm{e}^{i\delta_\star/2} \right\}.$$

(b) *If $a_\star \in (0,1)$, then $U_\star$ has purely absolutely continuous spectrum*

$$\sigma(U_\star) = \sigma_{\mathsf{ac}}(U_\star) = \left\{ \mathrm{e}^{i\gamma} \mid \gamma \in [\delta_\star/2 + \theta_\star, \pi + \delta_\star/2 - \theta_\star] \cup [\pi + \delta_\star/2 + \theta_\star, 2\pi + \delta_\star/2 - \theta_\star] \right\}$$

with $\theta_\star := \arccos(a_\star)$.

(c) *If $a_\star = 1$, then $U_\star$ has purely absolutely continuous spectrum*

$$\sigma(U_\star) = \sigma_{\mathsf{ac}}(U_\star) = \mathbb{S}^1.$$

Idea of the proof.

(a) Let $\mathcal{K} := \mathrm{L}^2\left([0,2\pi), \frac{\mathrm{d}k}{2\pi}, \mathbb{C}^2\right)$ and $\mathscr{F} : \mathcal{H} \to \mathcal{K}$ the Fourier transform

$$(\mathscr{F}\Psi)(k) := \sum_{x\in\mathbb{Z}} \mathrm{e}^{-ikx}\,\Psi(x), \quad \Psi \in \mathcal{H},\ k \in [0,2\pi).$$

For $f \in \mathcal{K}$ and a.e. $k \in [0,2\pi)$

$$(\mathscr{F}\,U_\star\,\mathscr{F}^*f)(k) = \widehat{U_\star}(k)f(k) \quad \text{with} \quad \widehat{U_\star}(k) := \begin{pmatrix} \mathrm{e}^{ik} & 0 \\ 0 & \mathrm{e}^{-ik} \end{pmatrix} C_\star \in \mathrm{U}(2),$$

and the claim follows by solving for $\lambda_{\star,j}(k) \in \mathbb{S}^1$ the characteristic equation

$$\det\left(\widehat{U_\star}(k) - \lambda_{\star,j}(k)\right) = 0, \quad j = 1,2,\ \text{a.e. } k \in [0,2\pi).$$

(b)-(c) With similar arguments we prove all the claims, except that $U_\star$ has purely absolutely continuous spectrum.

Idea of the proof (continued).

For this, we use Mourre theory with conjugate operator

$$A_\star\Psi := \tfrac{1}{2}\big(X_\star V_\star + V_\star X_\star\big)\Psi, \quad \Psi \in \mathcal{D}(A_\star) := \big\{\Psi \in \mathcal{H} \mid V_\star\Psi \in \mathcal{D}(X_\star)\big\}.$$

$V_\star := \mathscr{F}^* \widehat{V_\star} \mathscr{F} \in \mathscr{B}(\mathcal{H})$ is the velocity operator for $U_\star$ given by

$$\big(\widehat{V_\star} f\big)(k) := \widehat{V_\star}(k) f(k), \quad f \in \mathcal{K}, \text{ a.e. } k \in [0, 2\pi),$$

where

$$\begin{cases} \widehat{V_\star}(k) := \sum_{j=1}^{2} v_{\star,j}(k)\,\Pi_{\star,j}(k) \\ v_{\star,j}(k) := \frac{i\lambda'_{\star,j}(k)}{\lambda_{\star,j}(k)} \in \mathbb{R} \\ \Pi_{\star,j}(k) \in \mathscr{B}(\mathbb{C}^2) \text{ orthogonal projection for } \lambda_{\star,j}(k). \end{cases}$$

Idea of the proof (continued).

$X_\star := \mathscr{F}^*\widehat{X_\star}\mathscr{F}$ with $\widehat{X_\star}$ a first order differential operator.

The coefficients of $\widehat{X_\star}$ are chosen so that $U_\star \in C^2(A_\star)$ with

$$U_\star^{-1}[A_\star, U_\star] = V_\star^2.$$

This gives a Mourre estimate for $U_\star$ on $\sigma(U_\star) \setminus \partial\sigma(U_\star)$ because $V_\star^2$ is strictly positive on $\sigma(U_\star) \setminus \partial\sigma(U_\star)$.

Thus, the abstract theorem implies that $U_\star$ has purely absolutely continuous spectrum. □

Commutators in two Hilbert spaces

- U, unitary operator with spectral measure $E^U(\cdot)$ and spectrum $\sigma(U)$ in a Hilbert space $\mathcal{H}$
- U_0, auxiliary unitary operator with spectral measure $E^{U_0}(\cdot)$ and spectrum $\sigma(U_0)$ in an auxiliary Hilbert space $\mathcal{H}_0$
- A_0, self-adjoint operator in $\mathcal{H}_0$ with domain $\mathcal{D}(A_0)$
- $J \in \mathscr{B}(\mathcal{H}_0, \mathcal{H})$, identification operator

Theorem

Assume that

(i) $U_0 \in C^1(A_0)$ *in* $\mathcal{H}_0$,

(ii) *there is* $\mathscr{D} \subset \mathcal{D}(A_0 J^*) \subset \mathcal{H}$ *with* JA_0J^* *essentially self-adjoint on* $\mathscr{D}$, *and self-adjoint extension denoted by* A,

(iii) *compacity conditions relating* U, U_0, A_0 *and* J.

Then, if U_0 *satisfies a Mourre estimate with respect to* A_0 *on* Θ, U *also satisfies a Mourre estimate with respect to* A *on* Θ.

Thus, if $U \in C^{1+\varepsilon}(A)$ and U_0 satisfies a Mourre estimate with respect to A_0 on Θ, then U has at most finitely many eigenvalues in Θ (multiplicities counted), and U has no singular continuous spectrum in Θ.

Structure of the essential spectrum

To study the essential spectrum of the operator U of the quantum walk, we apply these commutator methods with auxiliary unitary operator

$$U_0 := U_\ell \oplus U_{\mathrm{r}} \quad \text{in} \quad \mathcal{H}_0 := \mathcal{H} \oplus \mathcal{H}.$$

The operator U_0 encodes the information of U_ℓ and U_{r}.

As intuition suggests, we have:

Theorem (Essential spectrum of U)

$$\sigma_{\mathrm{ess}}(U) = \sigma(U_0) = \sigma(U_\ell) \cup \sigma(U_{\mathrm{r}}).$$

Idea of the proof.

U can be represented in a crossed product C^*-algebra $\mathcal{A} \subset \mathscr{B}(\mathcal{H})$, and the image of U in the quotient algebra $\mathcal{A}/\mathscr{K}(\mathcal{H})$ is equal to U_0.

Thus,

$$\sigma_{\mathrm{ess}}(U) = \sigma(U_0) = \sigma(U_\ell) \cup \sigma(U_{\mathrm{r}}).$$

□

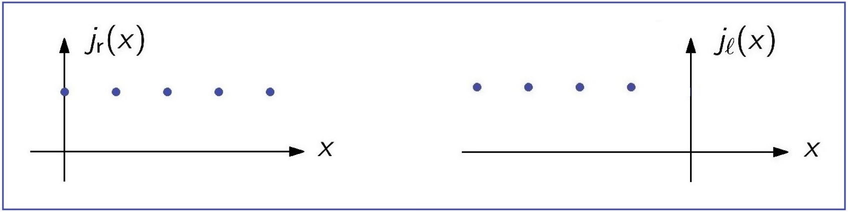
What about the nature of $\sigma_{\text{ess}}(U)$?

We define the identification operator $J \in \mathscr{B}(\mathcal{H}_0, \mathcal{H})$ as

$$J(\Psi_\ell, \Psi_\mathrm{r}) := j_\ell \Psi_\ell + j_\mathrm{r} \Psi_\mathrm{r}, \quad (\Psi_\ell, \Psi_\mathrm{r}) \in \mathcal{H}_0,$$

with

$$j_\mathrm{r}(x) := \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x \leq -1 \end{cases} \quad \text{and} \quad j_\ell := 1 - j_\mathrm{r}.$$



We choose $A_0 := A_\ell \oplus A_\mathrm{r}$ in $\mathcal{H}_0$ as conjugate operator for U_0.

- Since $U_\star \in C^2(A_\star)$ with $U_\star^{-1}[A_\star, U_\star] = V_\star^2$, we have

  $$U_0 \in C^2(A_0) \quad \text{with} \quad U_0^{-1}[A_0, U_0] = V_0^2 \quad \text{and} \quad V_0 := V_\ell \oplus V_\mathrm{r}.$$

  This gives a Mourre estimate for U_0 on $\sigma(U_0) \setminus \{\partial\sigma(U_\ell) \cup \partial\sigma(U_\mathrm{r})\}$.

- Using Nelson's commutator theorem, we show that JA_0J^* is essentially self-adjoint on

  $$\mathscr{D} := \big\{\Psi \in \mathcal{H} \mid \Psi \text{ has finite support}\big\},$$

  with self-adjoint extension denoted by A.

- Using the short-range assumption, we show the compacity conditions relating U, U_0, A_0 and J, and the regularity condition $U \in C^{1+\varepsilon}(A)$ for $\varepsilon \in (0,1)$ with $\varepsilon \leq \min\{\varepsilon_\ell, \varepsilon_\mathrm{r}\}$.

  (some toroidal pseudodifferential calculus is needed)

Combining these results with the abstract theorem in the two-Hilbert spaces setting, we get :

Theorem (Spectrum of U)

For any closed set $\Theta \subset \mathbb{T} \setminus \{\partial\sigma(U_\ell) \cup \partial\sigma(U_{\mathsf{r}})\}$, the operator U has at most finitely many eigenvalues in Θ, each one of finite multiplicity, and U has no singular continuous spectrum in Θ.

The set

$$\tau(U) := \partial\sigma(U_\ell) \cup \partial\sigma(U_{\mathsf{r}})$$

is interpreted as the set of thresholds of U, in the same way the singleton $\{0\} = \partial[0, \infty)$ is interpreted as a threshold in the Schrödinger case.

Gracias !

References

- J. Asch, O. Bourget, and A. Joye. Spectral stability of unitary network models. Rev. Math. Phys., 2015
- C. Fernández, S. Richard, and R. Tiedra de Aldecoa. Commutator methods for unitary operators. J. Spectr. Theory, 2013
- S. Richard, A. Suzuki and R. Tiedra de Aldecoa. Quantum walks with an anisotropic coin I: spectral theory. Lett. Math. Phys, 2018
- A. Suzuki. Asymptotic velocity of a position-dependent quantum walk. Quantum Inf. Process., 2016